This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

2/

1831 e. 11

Le Thomas

Le Thomas

London

1817

600047092S

/618

Thomas Carmart

ELEMENTS

The Gift of Dr Thomas

OF

of Professor in the Cou

GEOMETRY.

of Carmarthen

CONTAINING THE

To Trefecca College May 155

PRINCIPAL PROPOSITIONS

IN THE

15

FIRST SIX, AND THE ELEVENTH AND TWELFTH BOOKS

OF

EUCLID.

WITH NOTES CRITICAL AND EXPLANATORY.

By JOHN BONNYCASTLE,

OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

THE THIRD EDITION.

LONDON:
PRINTED FOR J. JOHNSON, NO. 72, ST. PAUL'S CHURCH-YARD,
BY BYE AND LAW, ST. JOHN'S-SQUARE, CLERKENWELL.

1803.

BODLEIAN
P
25 MAY 1965
LIBRARY

PREFACE.

OF all the works of antiquity which have been tranſmitted to the preſent times, none are more univerſally and deſervedly eſteemed than the Elements of Geometry which go under the name of EUCLID. In many other branches of ſcience the moderns have far ſurpaſſed their maſters; but, after a lapſe of more than two thouſand years, this performance ſtill maintains its original pre-eminence, and has even acquired additional celebrity from the fruitleſs attempts which have been made to eſtabliſh a different ſyſtem.

It is, however, generally allowed, that the Elements, as they now ſtand, are attended with many difficulties, which greatly retard the progreſs of learners, on their firſt entrance upon this ſtudy, and prevent them from applying to other branches of knowledge, which, in the preſent advanced ſtate of the ſciences, are equally uſeful and important. Among other obſtacles of this kind may be mentioned the theory of parallel lines, the doctrine of proportion, and many things in the eleventh and twelfth books, relating to ſolids, which are uſually found extremely embarraſſing; and notwithſtanding the numberleſs efforts which have been made to elucidate and explain them, are ſtill liable to many objections.

On this account, it has been found neceſſary, in moſt of our academical inſtitutions, to have recourſe to ſome of the more compendious rudiments of later writers, who, by means of a different arrangement, have endeavoured to new-model the ſubject, and to render it leſs complex and elaborate. But the greater part of them are ſo ill digeſted that they ſerve rather to miſlead the learner than to afford him any aſſiſtance. For, beſides being deficient in order and method, ſome of theſe authors have treated the ſubject algebraically; and others, by introducing a number of exceptionable principles, and a vague unſatisfactory mode of demonſtration, have degraded the ſcience, and deprived it of ſome of its moſt ſtriking advantages.

It is, therefore, the deſign of the following performance, to obviate theſe objections; and to render the ſubject more familiar and perſpicuous, without weakening its evidence, or deſtroying its elegance and ſimplicity. For this purpoſe, many propoſitions in EUCLID, which are of little or no uſe in their application, and were only introduced into the Elements as neceſſary links in the chain of reaſoning, are here omitted; and others ſubſtituted in their place, which are equally conducive to that end, and at the ſame time more uſeful and conciſe. By this means all the moſt eſſential principles of the ſcience have been brought into a ſhorter compaſs, and the demonſtrations, which lead to its ſublimer truths, ſo continued, as to render their connection as obvious and comprehenſive as poſſible.

Great care has alſo been taken to preſerve that methodical preciſion and rigour of proof, which, in treating of this ſubject, are requiſites of nearly equal importance with the ſcience itſelf. For independently of its other advantages, Geometry has always been conſidered as an excellent logic, which in forming the mind, and eſtabliſhing a habit of cloſe thinking and juſt reaſoning, in every enquiry after truth, is far ſuperior to all the dialectical principles that have yet been invented; the ſimplicity of its firſt principles; the clearneſs and certainty of its demonſtrations; the regular concatenation of its parts; and the univerſality of its application being ſuch as no other ſubject can boaſt.

For theſe reaſons, it was judged neceſſary to adhere as cloſely as poſſible to the plan of the original Elements; this being, in many reſpects, much more natural and judicious than any of thoſe which have ſince been propoſed by other writers. But as the work was rather deſigned as a regular Inſtitution of the moſt uſeful principles of the ſcience, than a ſtrict abridgment of EUCLID, ſome alterations have been made, both in the arrangement of the propoſitions and the mode of demonſtration; the latter of which, in particular, it is preſumed, will be found conſiderably improved, being here delivered in a more convenient form, and rendered as clear and explicit as the nature of the ſubject would admit.

In the firſt ſix books, every thing has been demonſtrated with a ſcrupulous accuracy; and it was at firſt deſigned that the ſame method ſhould have been obſerved throughout; but this, in treating of the ſolids, was found incompatible with the plan of the work, it being here ſcarcely poſſible to follow the ſtrict principles of EUCLID without becoming prolix and obſcure. It was therefore thought proper, in this part of the performance, to adopt a mode of proof, which though not geometrically exact, is far more perſpicuous than the former, and equally ſatisfactory and convincing to the mind; eſpecially in the way it is here given, which is ſomething leſs exceptionable than that of CAVALERIUS, by whom it was firſt introduced.

Many other particulars might be mentioned, in which this performance will be found to differ from moſt others of the like nature; but as they conſiſt chiefly of improvements and emendations which are too obvious to eſcape the notice of the reader, any further account of them would be unneceſſary. It is ſufficient to obſerve that much time and

2/

1831 e. 41

De Thomas

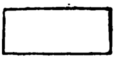
5

De Thomas

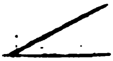
London

1817

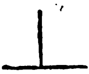
6. A plane ſuperficies is that which is every where perfectly flat and even.



7. A plain rectilineal angle is the inclination or opening of two right lines which meet in a point.

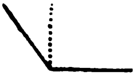


8. One right line is ſaid to be perpendicular to another, when it makes the angles on both ſides of it equal to each other.

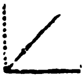


9. A right angle is that which is made by two right lines that are perpendicular to each other.

10. An obtuſe angle is that which is greater than a right angle.

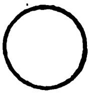


11. An acute angle is that which is leſs than a right angle.



12. A figure is that which is incloſed by one or more boundaries.

13. A circle is a plane figure, contained by one line, called the circumference, which is every where equally diſtant from a point within the figure, called the centre.



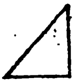
14. Rectilineal figures are thoſe which are contained by right lines.

15. All plane figures, bounded by three right lines, are called triangles.

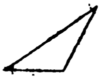
16. An equilateral triangle, is that which has all its ſides equal to each other.

17. An iſoſceles triangle, is that which has only two of its ſides equal to each other.

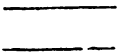
18. A right-angled triangle, is that which has one right angle; the ſide which is oppoſite to the right angle being called the hypothenuſe.



19. An obtuſe-angled triangle, is that which has one obtuſe angle.



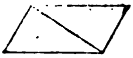
20. Parallel right lines are ſuch as are in the ſame plane, and which, being produced ever ſo far both ways, will never meet.



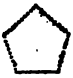
21. Every plane figure, bounded by four right lines, is called a quadrangle, or quadrilateral.

22. A parallelogram, is a quadrangle whoſe oppoſite ſides are parallel.

23. The diagonal of a quadrangle, is a right line joining any two of its oppoſite angles.



24. The baſe of any figure is that ſide upon which it is ſuppoſed to ſtand; and the vertical angle is that which is oppoſite to the baſe.



Note, When an angle is expreſſed by means of three letters, the one which ſtands at the angular point, muſt always be placed in the middle.

POSTULATES.

1. Let it be granted that a right line may be drawn from any one given point to another.

2. That a terminated right line, may be produced to any length in a right line.

3. That a circle may be deſcribed from any point as a centre, at any diſtance from that centre.

4. And that a right line, which meets one of two parallel right lines, may be produced till it meets the other.

AXIOMS.

1. Things which are equal to the ſame thing are equal to each other.

2. If equals be added to equals the wholes will be equal.

3. If equals be taken from equals the remainders will be equal.

4. If equals be added to unequals the wholes will be unequal.

5. If equals be taken from unequals the remainders will be unequal.

6. Things which are double of the ſame thing are equal to each other.

7. Things which are halves of the ſame thing are equal to each other.

8. The whole is equal to all its parts taken together.

9. Magnitudes which coincide, or fill the ſame ſpace, are equal to each other.

REMARKS.

A PROPOSITION, is ſomething which is either propoſed to be done, or to be demonſtrated.

A PROBLEM, is ſomething which is propoſed to be done.

A THEOREM, is ſomething which is propoſed to be demonſtrated.

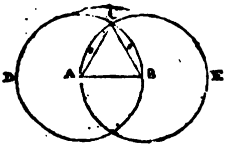
A LEMMA, is ſomething which is previouſly demonſtrated, in order to render what follows more eaſy.

A COROLLARY, is a conſequent truth, gained from ſome preceding truth, or demonſtration.

A SCHOLIUM, is a remark or obſervation made upon ſomething going before it.

PROPOSITION I. PROBLEM.

UPON a given finite right line to deſcribe an equilateral triangle.



Let AB be the given right line; it is required to deſcribe an equilateral triangle upon it.

From the point A, at the diſtance AB, deſcribe the circle BCD (*Poſ.* 3.)

And from the point B, at the diſtance BA, deſcribe the circle ACE (*Poſ.* 3.)

Then, becauſe the two circles paſs through each other's centres, they will cut each other.

And, if the right lines CA, CB be drawn from the point of interſection C, ABC will be the equilateral triangle required.

For, ſince A is the centre of the circle BCD, AC is equal to AB (*Def.* 13.)

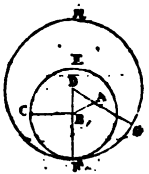
And, becauſe B is the centre of the circle ACE, BC is alſo equal to AB (*Def.* 13.)

But things which are equal to the ſame thing are equal to each other (*Ax.* 1); therefore AC is equal to CB.

And, ſince AC, CB are equal to each other, as well as to AB, the triangle ABC is equilateral; and it is deſcribed upon the right line AB, as was to be done.

PROP. II. PROBLEM.

From a given point to draw a right line equal to a given finite right line.



Let A be the given point, and BC the given right line; it is required to draw a right line from the point A, that ſhall be equal to BC.

Join the points A, B, (*Poſ.* 1.); and upon BA deſcribe the equilateral triangle BAD (*Prop.* 1.)

From the point B, at the diſtance BC, deſcribe the circle CEF (*Poſ.* 3.) cutting DB produced in F.

And from the point D, at the diſtance DF, deſcribe the circle FHG (*Poſ.* 3.); then, if DA be produced to G, AG will be equal to BC, as was required.

For, ſince B is the centre of the circle CEF, BC is equal to BF (*Def.* 13.)

And, becauſe D is the centre of the circle FHG, DG is equal to DF (*Def.* 13.)

But the part DA is alſo equal to the part DB (*Def.* 16.), whence the remainder AG will be equal to the remainder BF (*Ax.* 3.)

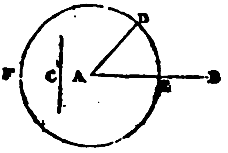
And ſince AG, BC have been each proved to be equal to BF, AG will alſo be equal to BC (*Ax.* 1.)

A right line AG, has, therefore, been drawn from the point A, equal to the right line BC, as was to be done.

SCHOLIUM. When the point A is at one of the extremities B, of the given line BC, any right line, drawn from that point to the circumference of the circle CEF, will be the one required.

PROP. III. PROBLEM.

From the greater of two given right lines, to cut off a part equal to the leſs.



Let AB and C be the two given right lines; it is required to cut off a part from AB, the greater, equal to C the leſs.

From the point A draw the right line AD equal to C (*Prop.* 2.); and from the centre A, at the diſtance AD, deſcribe the circle DEF (*Poſ.* 3.) cutting AB in E, and AE will be equal to C as was required.

For, ſince A is the centre of the circle EDF, AE will be equal to AD (*Def.* 13.)

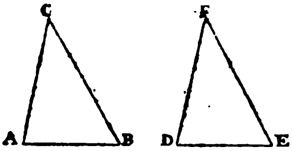
But C is equal to AD, by conſtruction; therefore AE will alſo be equal to C (*Ax.* 1.)

Whence, from AB, the greater of the two given lines, there has been taken a part equal to C the leſs, which was to be done.

SCHOLIUM. When the two given lines are ſo ſituate, that one of the extremities of C falls in the point A, the former part of the conſtruction becomes unneceſſary.

PROP. IV. THEOREM.

If two ſides and the included angle of one triangle, be equal to two ſides and the included angle of another, each to each, the triangles will be equal in all reſpects.



Let ABC, DEF be two triangles, having CA equal to FD, CB to FE, and the angle C to the angle F; then will the two triangles be equal in all reſpects.

For conceive the triangle ABC to be applied to the triangle DEF, ſo that the point C may coincide with the point F, and the ſide CA with the ſide FD.

Then, becauſe CA coincides with FD, and the angle C is equal to the angle F (*by Hyp.*), the ſide CB will alſo coincide with the ſide FE.

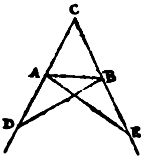
And, ſince CA is equal to FD, and CB to FE (*by Hyp.*), the point A will fall upon the point D, and the point B upon the point E.

But right lines, which have the ſame extremities, muſt coincide, or otherwiſe their parts would not lie in the ſame direction, which is abſurd (*Def.* 5.); therefore AB falls upon, and is equal to DE.

And, becauſe the triangle ABC coincides with the triangle DEF, the angle A will be equal to the angle D, the angle B to the angle E, and the two triangles will be equal in all reſpects (*Ax.* 9.) Q. E. D.

PROP. V. THEOREM.

The angles at the baſe of an iſoſceles triangle are equal to each other.



Let ABC be an iſoſceles triangle, having the ſide CA equal to the ſide CB; then will the angle CAB be equal to the angle CBA.

For, in CA and CB produced, take any two equal parts CD, CE (*Prop.* 3.), and join the points AE, BD:

Then, becauſe the two ſides CA, CE of the triangle CAE, are equal to the two ſides CB, CD of the triangle CBD, and the angle C is common, the ſide AE will alſo be equal to the ſide BD, the angle CAE to the angle CBD, and the angle D to the angle E (*Prop.* 4.)

And ſince the whole CD is equal to the whole CE (*by Conſt.*), and the part CA to the part CB (*by Hyp.*), the remaining part AD will alſo be equal to the remaining part BE (*Ax.* 3.)

Hence, the two ſides DA, DB, of the triangle DAB, being equal to the two ſides EB, EA of the triangle EBA, and the angle D to the angle E, the angle ABD will alſo be equal to the angle BAE (*Prop.* 4.)

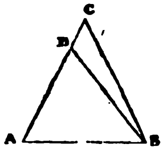
And if from the equal angles CAE, CBD, there be taken the equal angles BAE, ABD, the remaining angle

CAB will be equal to the remaining angle CBA (*Ax.* 3.) Q. E. D.

COROLLARY. Every equilateral triangle is alſo equiangular.

PROP. VI. THEOREM.

If two angles of a triangle be equal to each other, the ſides which are oppoſite to them will alſo be equal.



Let ABC be a triangle, having the angle CAB equal to the angle CBA; then will the ſide CA be equal to the ſide CB.

For if CA be not equal to CB, one of them muſt be greater than the other; let CA be the greater, and make AD equal to BC (*Prop.* 3.) and join BD.

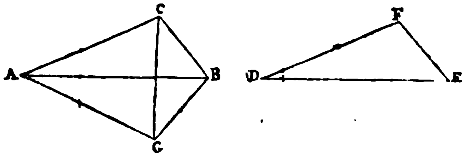
Then, becauſe the two ſides AD, AB, of the triangle ADB, are equal to the two ſides BC, BA, of the triangle ACB, and the angle DAB is equal to the angle CBA (*by Hyp.*), the triangle ADB will be equal to the triangle ACB (*Prop.* 4.), the leſs to the greater, which is abſurd.

The ſide CA, therefore, cannot be greater than the ſide CB; and, in the ſame manner, it may be ſhewn that it cannot be leſs; conſequently they are equal to each other. Q. E. D.

COROL. Every equiangular triangle is alſo equilateral.

PROP. VII. THEOREM.

If the three ſides of one triangle be equal to the three ſides of another, each to each, the angles which are oppoſite to the equal ſides will alſo be equal.



Let ABC, DEF be two triangles, having the ſide AB equal to the ſide DE, AC to DF, and BC to EF; then will the angle ACB be equal to the angle DFE, BAC to EDF, and ABC to DEF.

For, let the triangle DFE be applied to the triangle ACB, ſo that their longeſt ſides, DE, AB, may coincide, and the point F fall at G; and join CG.

Then, ſince the ſide AC is equal to the ſide DF, or AG (*by Hyp.*), the angle ACG will be equal to the angle AGC (*Prop.* 5.)

And, becauſe the ſide BC is equal to the ſide EF, or BG (*by Hyp.*), the angle BCG will be equal to the angle BGC (*Prop.* 5.)

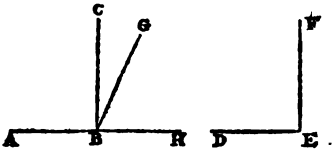
But ſince the angles ACG, BCG are equal to the angles AGC, BGC, each to each, the whole angle ACB will be equal to the whole angle AGB (*Ax.* 8.)

And, becauſe AC is equal to AG, BC to BG, and the angle ACB to the angle AGB, the angle CAB, will, alſo, be equal to the angle GAB, and the angle ABC to the angle ABG (*Prop.* 4.)

But the triangles AGB, DFE, are identical; conſequently the angles of the triangle DFE will, alſo, be equal to the correſponding angles of the triangle ACB. Q.E.D.

PROP. VIII. THEOREM.

All right angles are equal to each other.



Let ABC, DEF be each of them right angles; then will ABC be equal to DEF.

For conceive the angle DEF to be applied to the angle ABC, ſo that the point E may coincide with the point B, and the line ED with the line BA.

And if EF does not coincide with BC, let it fall, if poſſible, without the angle ABC, in the direction BG; and produce AB to H.

Then, becauſe the angles ABC, ABG are right angles (*by Hyp.*), the lines CB, GB will be each perpendicular to AH (*Def.* 8. 9.)

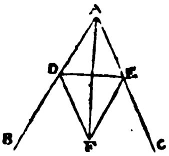
And, ſince a right line which is perpendicular to another right line, makes the angles on each ſide of it equal (*Def.* 8.), the angle CBA will be equal to the angle CBH, and the angle GBA to the angle GBH.

But the angle GBA is greater than the angle CBA, or its equal CBH; conſequently the angle GBH is alſo greater than the angle CBH; that is, a part is greater than the whole, which is abſurd.

The line EF, therefore, does not fall without the angle ABC: and in the ſame manner it may be ſhewn that it does not fall within it; conſequently EF and BC will coincide, and the angle DEF be equal to the angle ABC, as was to be ſhewn.

PROP. IX. PROBLEM.

To biſect a given rectilineal angle, that is, to divide it into two equal parts.



Let BAC be the given rectilineal angle; it is required to divide it into two equal parts.

Take any point D in AB, and from AC cut off AE equal to AD (*Prop.* 3.), and join DE.

Upon DE deſcribe the equilateral triangle DFE (*Prop.* 1.), and join AF; then will AF biſect the angle BAC, as was required.

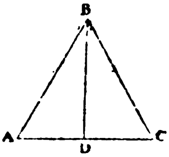
For AD is equal to AE, by conſtruction; DF is alſo equal to FE (*Def.* 16.), and AF is common to each of the triangles AFD, AFE.

But when the three ſides of one triangle are equal to the three ſides of another, each to each, the angles which are oppoſite to the equal ſides are, alſo, equal (*Prop.* 7.)

The ſide DF, therefore, being equal to the ſide FE, the angle DAF will be equal to the angle FAE; and conſequently the angle BAC is biſected by the right line AF, as was to be done.

PROP. X. PROBLEM.

To biſect a given finite right line, that is, to divide it into two equal parts.



Let AC be the given right line; it is required to divide it into two equal parts.

Upon AC deſcribe the equilateral triangle ACB (*Prop.* 1.), and biſect the angle ABC by the right line BD (*Prop.* 9.); then will AC be divided into two equal parts at the point D, as was required.

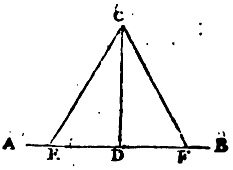
For AB is equal to BC (*Def.* 16.), BD is common to each of the triangles ADB, CDB, and the angle ABD is equal to the angle CBD (*by Conſt.*)

But when two ſides and the included angle of one triangle, are equal to two ſides and the included angle of another, each to each, their baſes will alſo be equal (*Prop.* 4.)

The baſe AD is, therefore, equal to the baſe DC; and, conſequently, the right line AC is biſected in the point D, as was to be done.

PROP. XI. PROBLEM.

At a given point, in a given right line, to erect a perpendicular.



Let AB be the given right line; and D the given point in it; it is required to draw a right line, from the point D, that ſhall be perpendicular to AB.

Take any point E, in AB, and make DF equal to DE (*Prop.* 3.); and upon EF deſcribe the equilateral triangle ECF (*Prop*: 1.)

Join the points D, C; and the right line CD will be perpendicular to AB, as was required.

For CE is equal to CF (*Def.* 16.), ED to DF (*by Conſt.*) and CD is common to each of the triangles ECD, FCD.

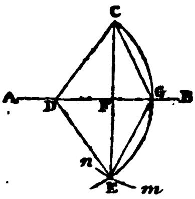
The three ſides of the triangle ECD being, therefore, equal to the three ſides of the triangle FCD, each to each; the angle EDC will, alſo, be equal to the angle FDC (*Prop.* 7.)

But one right line is perpendicular to another when the angles on both ſides of it are equal (*Def.* 8.); therefore CD is perpendicular to AB; and it is drawn from the point D as was to be done.

SCHOLIUM. If the given point be at, or near, the end of AB, the line muſt be produced.

PROP. XII. PROBLEM.

To draw a right line perpendicular to a given right line, of an unlimited length, from a given point without it.



Let AB be the given right line, and C the given point; it is required to draw a right line from the point C, that ſhall be perpendicular to AB.

Take any point D, in AB, and from that point, with the diſtance DC, deſcribe the circle CGE, cutting AB in G.

Join GC, and from the point G, with the diſtance GC, deſcribe the circle *n* E *m*, cutting the former in E.

Through the points C, E draw the right line CFE, cutting AB in F, and CF will be perpendicular to AB, as was required.

For, join the points D, C, D, E, and G, E:

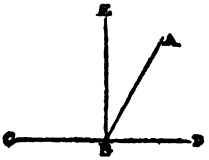
Then, becauſe DC is equal to DE, GC to GE (*Def.* 13.) and DG common to each of the triangles DCG, DEG, the angle CDG will be equal to the angle GDE (*Prop.* 7.)

And, ſince DC is equal to DE, DF common to each of the triangles DCF, DEF, and the angle CDG equal to the angle GDE, the angle DFC will alſo be equal to the angle DFE (*Prop.* 4.)

But one line is perpendicular to another when the angles on both ſides of it are equal (*Def.* 8); therefore CF is perpendicular to AB; and it is drawn from the point C, as was to be done.

PROP. XIII. THEOREM.

The angles which one right line makes with another, on the ſame ſide of it, are together equal to two right angles.



Let the right line AB fall upon the right line CD; then will the angles ABC, ABD, taken together, be equal to two right angles.

For if the angles ABC, ABD be equal to each other, they will be, each of them, right angles (*Def.* 8 *and* 9.);

But if they be unequal, let EB be drawn, from the point B, perpendicular to CD (*Prop.* 11.)

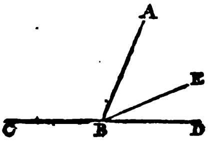
Then, ſince the angles EBC, EBD are right angles (*Def.* 8.), and the angle EBD is equal to the angles EBA, ABD (*Ax.* 8.), the angles EBC, EBA and ABD will be equal to two right angles.

But the angles EBC, EBA are, together, equal to the angle ABC (*Ax.* 8.); conſequently the angles ABC, ABD are, alſo, equal to two right angles. Q. E. D.

COROLL. All the angles which can be made, at any point B, on the ſame ſide of the right line CD, are, together, equal to two right angles.

PROP. XIV. THEOREM.

If a right line meet two other right lines, in the ſame point, and make the angles on each ſide of it together equal to two right angles, thoſe lines will form one continued right line.



Let the right line AB meet the two right lines CB, BD, at the point B, and make the angles ABC, ABD together equal to two right angles, then will BD be in the ſame right line with CB.

For, if it be not, let ſome other line BE be in the ſame right line with CB.

Then, becauſe the right line AB falls upon the right line CBE, the angles ABC, ABE, taken together, are equal to two right angles (*Prop.* 13.)

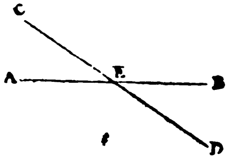
But the angles ABC, ABD are alſo equal to two right angles (*by Hyp.*); conſequently the angles ABC, ABE are equal to the angles ABC, ABD.

And, if the angle ABC, which is common, be taken away, the remaining angle ABE will be equal to the remaining angle ABD; the leſs to the greater, which is abſurd.

The line BE, therefore, is not in the ſame right line with CB; and the ſame may be proved of any other line but BD; conſequently CBD is one continued right line, as was to be ſhewn.

PROP. XV. THEOREM.

If two right lines interſect each other, the oppoſite angles will be equal.



Let the two right lines AB, CD interſect each other in the point E; then will the angle AEC be equal to the angle DEB, and the angle AED to the angle CEB.

For, ſince the right line CE falls upon the right line AB, the angles AEC, CEB, taken together, are equal to two right angles (*Prop.* 13.)

And, becauſe the right line BE falls upon the right line CD, the angles BED, CEB, taken together, are alſo equal to two right angles (*Prop.* 13.)

The angles AEC, CEB, taken together, are, therefore, equal to the angles BED, CEB taken together (*Ax.* 1.)

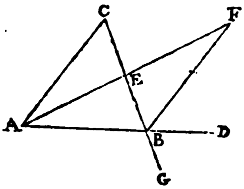
And, if the angle CEB, which is common, be taken away, the remaining angle AEC will be equal to the remaining angle BED (*Ax.* 3.)

And, in the ſame manner, it may be ſhewn that the angle AED is equal to the angle CEB. Q. E. D.

COROLL. All the angles made by any number of right lines, meeting in a point, are together equal to four right angles.

PROP. XVI. THEOREM.

If one ſide of a triangle be produced, the outward angle will be greater than either of the inward oppoſite angles.



Let ABC be a triangle, having the ſide AB produced to D; then will the outward angle CBD be greater than either of the inward oppoſite angles BAC or ACB.

For, biſect BC in E (*Prop.* 10.), and join AE; in which, produced, take EF equal to AE (*Prop.* 3.), and join BF.

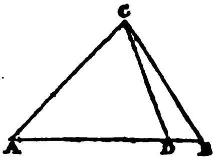
Then, ſince AE is equal to EF, EC to EB (*by Conſt.*) and the angle AEC to the angle BEF (*Prop.* 15.), the angle ACE will, alſo, be equal to the angle EBF (*Prop.* 4.)

But the angle CBD is greater than the angle EBF; conſequently it is alſo greater than the angle ACE.

And, if CB be produced to G, and AB be biſected, it may be ſhewn, in like manner, that the angle ABG, or its equal CBD, is greater than CAB. Q. E. D.

PROP. XVII. THEOREM.

The greater ſide of every triangle is oppoſite to the greater angle; and the greater angle to the greater ſide.



Let ABC be a triangle, having the ſide AB greater than the ſide AC; then will the angle ACB be greater than the angle ABC.

For, ſince AB is greater than AC, let AD be taken equal to AC (*Prop.* 3.), and join CD.

Then, ſince CDB is a triangle, the outward angle ADC is greater than the inward oppoſite angle DBC (*Prop.* 16.)

But the angle ACD is equal to the angle ADC, becauſe AC is equal to AD; conſequently the angle ACD is, alſo, greater than DBC or ABC.

And, ſince ACD is only a part of ACB, the whole angle ACB muſt be much greater than the angle ABC.

Again, let the angle ACB be greater than the angle ABC, then will the ſide AB be greater than the ſide AC.

For, if AB be not greater than AC, it muſt be either equal or leſs.

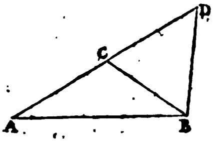
But it cannot be equal, for then the angle ACB would be equal to the angle ABC (*Prop.* 5.), which it is not.

Neither can it be leſs, for then the angle ACB would be leſs than the angle ABC (*Prop.* 17.), which it is not.

The ſide AB, therefore, is neither equal to AC, nor leſs than it; conſequently it muſt be greater. Q. E. D.

PROP. XVIII. THEOREM.

Any two ſides of a triangle, taken together, are greater than the third ſide.



Let ABC be a triangle; then will any two ſides of it, taken together, be greater than the third ſide.

For, in AC produced, take CD equal to CB (*Prop.* 3.), and join BD.

Then, becauſe CD is equal to CB (*by conſt.*), the angle CDB will be equal to the angle CBD (*Prop.* 5.)

But the angle ABD is greater than the angle CBD, conſequently it muſt alſo be greater than the angle ADB.

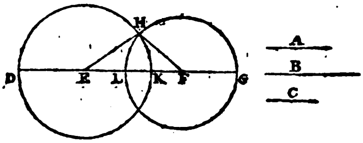
And, ſince the greater ſide of every triangle, is oppoſite to the greater angle (*Prop.* 17.), the ſide AD is greater than the ſide AB.

But AD is equal to AC and CB taken together (*by conſt.*); therefore AC, CB are alſo greater than AB.

And, in the ſame manner, it may be ſhewn, that any other two ſides, taken together, are greater than the third ſide. Q. E. D.

PROP. XIX. Problem.

To deſcribe a triangle, whoſe ſides ſhall be equal to three given right lines, provided any two of them, taken together, be greater than the third.



Let A, B, C be the three given right lines, any two of which, taken together, are greater than the third; it is required to make a triangle whoſe ſides ſhall be equal to A, B, C reſpectively.

Draw any right line DG; on which take DE equal to A, EF equal to B, and FG equal to C (*Prop.* 3.)

From the point E, with the diſtance ED, deſcribe the circle KHD, cutting DG in K; and from the point F, with the diſtance FG, deſcribe the circle GHL, cutting DG in L.

Then, becauſe EG is greater than ED (*by Hyp.*), or its equal EK, the point G, which is in the circumference of the circle GHL, will fall without the circle KHD.

And, becauſe FD is greater than FG (*by Hyp.*), or its equal FL, the point D, which is in the circumference of the circle KHD, will fall without the circle GHL.

But ſince a part of the circle GHL falls without the circle KHD, and a part of the circle KHD falls without the circle GHL, neither of the circles can be included within the other.

Again, becauſe DE, FG, or their equals EK, FL are, together, greater than EF (*by Hyp.*), the two circles can neither touch nor fally without each other.

They muſt, therefore, cut one another, in ſome point H; and if the right lines EH, FH be drawn, EHF will be the triangle required.

For, ſince E is the centre of the circle KHD, EH is equal to ED (*Def.* 13.); but ED is equal to A (*by Conſt.*); therefore EH is alſo equal to A.

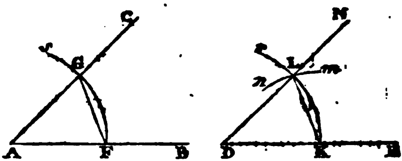
And, becauſe F is the centre of the circle GHL, FH is equal to FG (*Def.* 13.); but FG is equal to C (*by Conſt.*); therefore FH is alſo equal to C.

And ſince EF is, likewiſe, equal to B (*by Conſt.*), the three ſides of the triangle EHF are reſpectively equal to the three given lines A, B, C, which was to be ſhewn.

PROP.

PROP. XX. PROBLEM.

At a given point, in a given right line, to make a rectilineal angle equal to a given rectilineal angle.



Let DE be the given right line, D the given point, and BAC the given rectilineal angle; it is required to make an angle at the point D that ſhall be equal to BAC.

Take any point F in AB, and from the point A, at the diſtance AF, deſcribe the circle FG*r*, cutting AC in G; and join FG.

Make DK equal to AF, and KE equal to FG (*Prop.* 3.); and from the points D, K, at the diſtances DK, KE, deſcribe the circles KL*r* and *n*L*m*, cutting each other in L.

Through the points D, L draw the right line DN, and the angle EDN will be equal to BAC, as was required.

For, join KL: then ſince AG is equal to AF (*Def.* 13.), and AF is equal to DK (*by Conſt.*), AG will alſo be equal to DK (*Ax.* 1.)

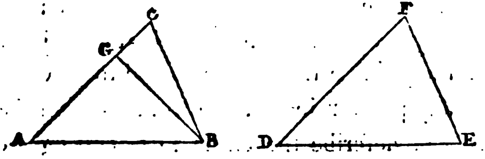
But DK is equal to DL (*Def.* 13.); conſequently AG is alſo equal to DL (*Ax.* 1.); and FG is equal to KE or KL (*by Conſt.*)

The three ſides of the triangle DKL are, therefore, equal to the three ſides of the triangle AFG, each to each;

whence the angle KDL is equal to the angle FAG, or BAC (*Prop.* 7.); and it is made at the point D, as was to be done.

PROP. XXI. THEOREM.

If two triangles be mutually equiangular, and have two correſponding ſides equal to each other, the other correſponding ſides will alſo be equal, and the two triangles will be equal in all reſpects.



Let the triangles ABC, DEF be mutually equiangular, and have the ſide AB equal to the ſide DE; then will the ſide AC be alſo equal to the ſide DF, the ſide BC to the ſide EF, and the two triangles will be equal in all reſpects.

For, if AC be not equal to DF, one of them muſt be greater than the other; let AC be the greater, and make AG equal to DF (*Prop.* 3.); and join BG.

Then, ſince the two ſides AB, AG, are equal to the two ſides DE, DF, each to each, and the angle GAB is equal to the angle FDE (*by Hyp.*), the angle ABG will, alſo, be equal to the angle DEF (*Prop.* 4.)

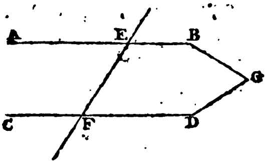
But the angle DEF is equal to the angle ABC (*by Hyp.*); conſequently the angle ABG will, alſo, be equal to the angle ABC, the leſs to the greater, which is abſurd.

The ſide AC, therefore, cannot be greater than the ſide DF; and, in the ſame manner, it may be ſhewn that it cannot be leſs; conſequently it muſt be equal to it.

And, ſince the two ſides AC, AB, are equal to the two ſides DF, DE, each to each, and the angle CAB is equal to the angle FDE, the ſide BC will alſo be equal to the ſide EF, and the two triangles will be equal in all reſpects (*Prop.* 4.) Q. E. D.

PROP. XXII. THEOREM.

If a right line interſect two other right lines, and make the alternate angles equal to each other, thoſe lines will be parallel.



Let the right line EF interſect the two right lines AB, CD, and make the alternate angles AEF, EFD equal to each other; then will AB be parallel to CD.

For, if they be not parallel, let them be produced, and they will meet each other, either on the ſide AC, or on the ſide BD (*Def.* 20.)

Suppoſe them to meet in the point G, on the ſide BD.

Then, ſince FGE is a triangle, the outward angle AEF is greater than the inward oppoſite angle EFD (*Prop.* 16.)

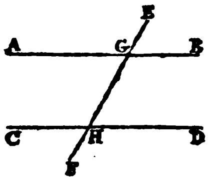
But the angles, AEF, EFD, are equal to each other (*by Hyp.*); whence they are equal and unequal at the ſame time, which is abſurd.

The lines AB, CD, therefore, cannot meet on the ſide BD; and, in the ſame manner, it may be ſhewn that they cannot meet on the ſide AC; conſequently they muſt be parallel to each other (*Def.* 20.) Q. E. D.

COROLL. Right lines which are perpendicular to the ſame right line are parallel to each other.

PROP. XXIII. THEOREM.

If a right line interſect two other right lines, and make the outward angle equal to the inward oppoſite one, on the ſame ſide; or the two inward angles, on the ſame ſide, together equal to two right angles, thoſe lines will be parallel.



Let the right line EF interſect the two right lines AB, CD, and make the outward angle EGB equal to the inward angle GHD; or the two inward angles BGH, GHD together equal to two right angles; then will AB be parallel to CD.

For, ſince the angles EGB, GHD are equal to each other (*by Hyp.*), and the angles AGH, EGB are alſo equal to each other (*Prop.* 15.), the angle AGH will be equal to the angle GHD (*Ax.* 1.)

But when a right line interſects two other right lines, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 22.); therefore AB is parallel to CD.

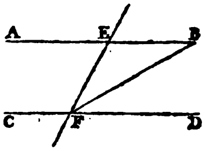
Again, ſince the angles BGH, GHD are, together, equal to two right angles (*by Hyp.*), and AGH, BGH are, alſo, equal to two right angles, (*Prop.* 13.) the angles AGH, BGH will be equal to the angles BGH, GHD (*Ax.* 1.)

And, if the common angle BGH be taken away, the remaining angle AGH will be equal to the remaining angle GHD (*Ax.* 3.)

But theſe are alternate angles: therefore, in this caſe, AB will, alſo, be parallel to CD (*Prop.* 22.) Q. E. D.

PROP. XXIV. THEOREM.

If a right line interſect two parallel right lines it will make the alternate angles equal to each other.



Let the right line EF interſect the two parallel right lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

For if they be not equal, one of them muſt be greater than the other; let EFD be the greater; and make the angle EFB equal to AEF (*Prop.* 20.)

Then, ſince AB, CD are parallel, the right line FB, which interſects CD, being produced, will meet AB in ſome point B (*Poſ.* 4.)

And, ſince EFB is a triangle, the outward angle AEF will be greater than the inward oppoſite angle EFB. (*Prop.* 16.)

But the angles AEF, EFB are equal to each other (*by Conſt.*) whence they are equal and unequal, at the ſame time, which is abſurd.

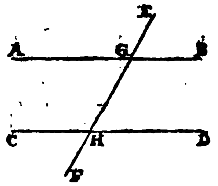
The angle EFD, therefore, is not greater than the angle AEF; and, in the ſame manner, it may be ſhewn that it is not leſs; conſequently they muſt be equal to each other. Q. E. D.

COROLL. Right lines which are perpendicular to one of two parallel right lines, are alſo perpendicular to the other.

PROP.

PROP. XXV. THEOREM.

If a right line interſect two parallel right lines, the outward angle will be equal to the inward oppoſite one, on the ſame ſide; and the two inward angles, on the ſame ſide, will be equal to two right angles.



Let the right line EF interſect the two parallel right lines AB, CD; then will the outward angle EGB be equal to the inward oppoſite angle GHD; and the two inward angles BGH, GHD will be equal to two right angles.

For, ſince the right line EF interſects the two parallel right lines AB, CD, the angle AGH will be equal to the alternate angle GHD (*Prop.* 24.)

But the angle AGH is equal to the oppoſite angle EGB (*Prop.* 15.); therefore the angle EGB will, alſo, be equal to the angle GHD.

Again, ſince the right line BG falls upon the right line EF, the angles EGB, BGH, taken together, are equal to two right angles (*Prop.* 13.)

But the angle EGB has been ſhewn to be equal to the angle GHD; therefore, the angles BGH, GHD, taken together, will, alſo, be equal to two right angles. Q.E.D.

For, if AB be not greater than AC, it muſt be either equal or leſs.

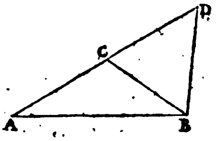
But it cannot be equal, for then the angle ACB would be equal to the angle ABC (*Prop.* 5.), which it is not.

Neither can it be leſs, for then the angle ACB would be leſs than the angle ABC (*Prop.* 17.), which it is not.

The ſide AB, therefore, is neither equal to AC, nor leſs than it; conſequently it muſt be greater. Q. E. D.

PROP. XVIII. THEOREM.

Any two ſides of a triangle, taken together, are greater than the third ſide.



Let ABC be a triangle; then will any two ſides of it, taken together, be greater than the third ſide.

For, in AC produced, take CD equal to CB (*Prop.* 3.), and join BD.

Then, becauſe CD is equal to CB (*by conſt.*), the angle CDB will be equal to the angle CBD (*Prop.* 5.)

But the angle ABD is greater than the angle CBD, conſequently it muſt alſo be greater than the angle ADB.

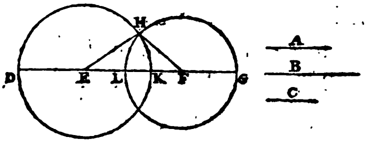
And, ſince the greater ſide of every triangle, is oppoſite to the greater angle (*Prop.* 17.), the ſide AD is greater than the ſide AB.

But AD is equal to AC and CB taken together (*by conſt.*); therefore AC, CB are alſo greater than AB.

And, in the ſame manner, it may be ſhewn, that any other two ſides, taken together, are greater than the third ſide. Q. E. D.

PROP. XIX. PROBLEM.

To deſcribe a triangle, whoſe ſides ſhall be equal to three given right lines, provided any two of them, taken together, be greater than the third.



Let A, B, C be the three given right lines, any two of which, taken together, are greater than the third; it is required to make a triangle whoſe ſides ſhall be equal to A, B, C reſpectively.

Draw any right line DG; on which take DE equal to A, EF equal to B, and FG equal to C (*Prop.* 3.)

From the point E, with the diſtance ED, deſcribe the circle KHD, cutting DG in K; and from the point F, with the diſtance FG, deſcribe the circle GHL, cutting DG in L.

Then, becauſe EG is greater than ED (*by Hyp.*), or its equal EK, the point G, which is in the circumference of the circle GHL, will fall without the circle KHD.

And, becauſe FD is greater than FG (*by Hyp.*), or its equal FL, the point D, which is in the circumference of the circle KHD, will fall without the circle GHL.

But

For, if AB be not greater than AC, it muſt be either equal or leſs.

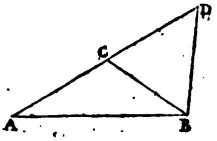
But it cannot be equal, for then the angle ACB would be equal to the angle ABC (*Prop.* 5.), which it is not.

Neither can it be leſs, for then the angle ACB would be leſs than the angle ABC (*Prop.* 17.), which it is not.

The ſide AB, therefore, is neither equal to AC, nor leſs than it; conſequently it muſt be greater. Q. E. D.

PROP. XVIII. THEOREM.

Any two ſides of a triangle, taken together, are greater than the third ſide.



Let ABC be a triangle; then will any two ſides of it, taken together, be greater than the third ſide.

For, in AC produced, take CD equal to CB (*Prop.* 3.), and join BD.

Then, becauſe CD is equal to CB (*by conſt.*), the angle CDB will be equal to the angle CBD (*Prop.* 5.)

But the angle ABD is greater than the angle CBD, conſequently it muſt alſo be greater than the angle ADB.

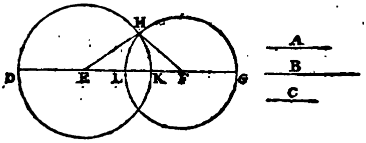
And, ſince the greater ſide of every triangle, is oppoſite to the greater angle (*Prop.* 17.), the ſide AD is greater than the ſide AB.

But AD is equal to AC and CB taken together (*by conſt.*); therefore AC, CB are alſo greater than AB.

And, in the ſame manner, it may be ſhewn, that any other two ſides, taken together, are greater than the third ſide. Q. E. D.

PROP. XIX. PROBLEM.

To deſcribe a triangle, whoſe ſides ſhall be equal to three given right lines, provided any two of them, taken together, be greater than the third.



Let A, B, C be the three given right lines, any two of which, taken together, are greater than the third; it is required to make a triangle whoſe ſides ſhall be equal to A, B, C reſpectively.

Draw any right line DG; on which take DE equal to A, EF equal to B, and FG equal to C (*Prop.* 3.)

From the point E, with the diſtance ED, deſcribe the circle KHD, cutting DG in K; and from the point F, with the diſtance FG, deſcribe the circle GHL, cutting DG in L.

Then, becauſe EG is greater than ED (*by Hyp.*), or its equal EK, the point G, which is in the circumference of the circle GHL, will fall without the circle KHD.

And, becauſe FD is greater than FG (*by Hyp.*), or its equal FL, the point D, which is in the circumference of the circle KHD, will fall without the circle GHL.

But ſince a part of the circle GHL falls without the circle KHD, and a part of the circle KHD falls without the circle GHL, neither of the circles can be included within the other.

Again, becauſe DE, FG, or their equals EK, FL are, together, greater than EF (*by Hyp.*), the two circles can neither touch nor fally without each other.

They muſt, therefore, cut one another, in ſome point H; and if the right lines EH, FH be drawn, EHF will be the triangle required.

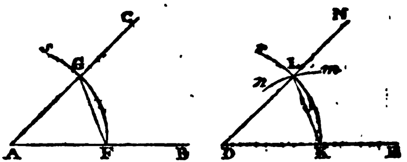
For, ſince E is the centre of the circle KHD, EH is equal to ED (*Def.* 13.); but ED is equal to A (*by Conſt.*); therefore EH is alſo equal to A.

And, becauſe F is the centre of the circle GHL, FH is equal to FG (*Def.* 13.); but FG is equal to C (*by Conſt.*); therefore FH is alſo equal to C.

And ſince EF is, likewiſe, equal to B (*by Conſt.*), the three ſides of the triangle EHF are reſpectively equal to the three given lines A, B, C, which was to be ſhewn.

PROP. XX. PROBLEM.

At a given point, in a given right line, to make a rectilineal angle equal to a given rectilineal angle.



Let DE be the given right line, D the given point, and BAC the given rectilineal angle; it is required to make an angle at the point D that ſhall be equal to BAC.

Take any point F in AB, and from the point A, at the diſtance AF, deſcribe the circle FGr, cutting AC in G; and join FG.

Make DK equal to AF, and KE equal to FG (*Prop.* 3.); and from the points D, K, at the diſtances DK, KE, deſcribe the circles KLr and *n*L*m*, cutting each other in L.

Through the points D, L draw the right line DN, and the angle EDN will be equal to BAC, as was required.

For, join KL: then ſince AG is equal to AF (*Def.* 13.), and AF is equal to DK (*by Conſt.*), AG will alſo be equal to DK (*Ax.* 1.)

But DK is equal to DL (*Def.* 13.); conſequently AG is alſo equal to DL (*Ax.* 1.); and FG is equal to KE or KL (*by Conſt.*)

The three ſides of the triangle DKL are, therefore, equal to the three ſides of the triangle AFG, each to each; whence

But ſince a part of the circle GHL falls without the circle KHD, and a part of the circle KHD falls without the circle GHL, neither of the circles can be included within the other.

Again, becauſe DE, FG, or their equals EK, FL are, together, greater than EF (*by Hyp.*), the two circles can neither touch nor fally without each other.

They muſt, therefore, cut one another, in ſome point H; and if the right lines EH, FH be drawn, EHF will be the triangle required.

For, ſince E is the centre of the circle KHD, EH is equal to ED (*Def.* 13.); but ED is equal to A (*by Conſt.*); therefore EH is alſo equal to A.

And, becauſe F is the centre of the circle GHL, FH is equal to FG (*Def.* 13.); but FG is equal to C (*by Conſt.*); therefore FH is alſo equal to C.

And ſince EF is, likewiſe, equal to B (*by Conſt.*), the three ſides of the triangle EHF are reſpectively equal to the three given lines A, B, C, which was to be ſhewn.

PROP.

For, ſince the angles EGB, GHD are equal to each other (*by Hyp.*), and the angles AGH, EGB are alſo equal to each other (*Prop.* 15.), the angle AGH will be equal to the angle GHD (*Ax.* 1.)

But when a right line interſects two other right lines, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 22.); therefore AB is parallel to CD.

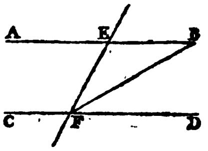
Again, ſince the angles BGH, GHD are, together, equal to two right angles (*by Hyp.*), and AGH, BGH are, alſo, equal to two right angles, (*Prop.* 13.) the angles AGH, BGH will be equal to the angles BGH, GHD (*Ax.* 1.)

And, if the common angle BGH be taken away, the remaining angle AGH will be equal to the remaining angle GHD (*Ax.* 3.)

But theſe are alternate angles: therefore, in this caſe, AB will, alſo, be parallel to CD (*Prop.* 22.) Q. E. D.

PROP. XXIV. THEOREM.

If a right line interſect two parallel right lines it will make the alternate angles equal to each other.

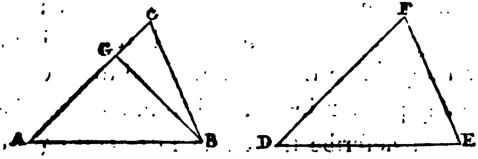


Let the right line EF interſect the two parallel right lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

whence the angle KDL is equal to the angle FAG, or BAC (*Prop.* 7.); and it is made at the point D, as was to be done.

PROP. XXI. THEOREM.

If two triangles be mutually equiangular, and have two correſponding ſides equal to each other, the other correſponding ſides will alſo be equal, and the two triangles will be equal in all reſpects.



Let the triangles ABC, DEF be mutually equiangular, and have the ſide AB equal to the ſide DE; then will the ſide AC be alſo equal to the ſide DF, the ſide BC to the ſide EF, and the two triangles will be equal in all reſpects.

For, if AC be not equal to DF, one of them muſt be greater than the other; let AC be the greater, and make AG equal to DF (*Prop.* 3.); and join BG.

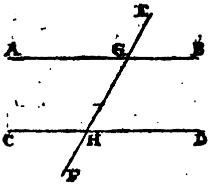
Then, ſince the two ſides AB, AG, are equal to the two ſides DE, DF, each to each, and the angle GAB is equal to the angle FDE (*by Hyp.*), the angle ABG will, alſo, be equal to the angle DEF (*Prop.* 4.)

But the angle DEF is equal to the angle ABC (*by Hyp.*); conſequently the angle ABG will, alſo, be equal to the angle ABC, the leſs to the greater, which is abſurd.

The

PROP. XXV. THEOREM.

If a right line interſect two parallel right lines, the outward angle will be equal to the inward oppoſite one, on the ſame ſide; and the two inward angles, on the ſame ſide, will be equal to two right angles.



Let the right line EF interſect the two parallel right lines AB, CD; then will the outward angle EGB be equal to the inward oppoſite angle GHD; and the two inward angles BGH, GHD will be equal to two right angles.

For, ſince the right line EF interſects the two parallel right lines AB, CD, the angle AGH will be equal to the alternate angle GHD (*Prop.* 24.)

But the angle AGH is equal to the oppoſite angle EGB (*Prop.* 15.); therefore the angle EGB will, alſo, be equal to the angle GHD.

Again, ſince the right line BG falls upon the right line EF, the angles EGB, BGH, taken together, are equal to two right angles (*Prop.* 13.)

But the angle EGB has been ſhewn to be equal to the angle GHD; therefore, the angles BGH, GHD, taken together, will, alſo, be equal to two right angles. Q.E.D.

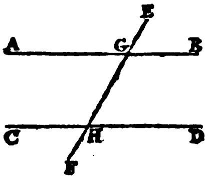
But the angles, AEF, EFD, are equal to each other (*by Hyp.*); whence they are equal and unequal at the ſame time, which is abſurd.

The lines AB, CD, therefore, cannot meet on the ſide ED; and, in the ſame manner, it may be ſhewn that they cannot meet on the ſide AC; conſequently they muſt be parallel to each other (*Def.* 20.) Q. E. D.

COROLL. Right lines which are perpendicular to the ſame right line are parallel to each other.

PROP. XXIII. THEOREM.

If a right line interſect two other right lines, and make the outward angle equal to the inward oppoſite one, on the ſame ſide; or the two inward angles, on the ſame ſide, together equal to two right angles, thoſe lines will be parallel.



Let the right line EF interſect the two right lines AB, CD, and make the outward angle EGB equal to the inward angle GHD; or the two inward angles BGH, GHD together equal to two right angles; then will AB be parallel to CD.

For, ſince the angles EGB, GHD are equal to each other (*by Hyp.*), and the angles AGH, EGB are alſo equal to each other (*Prop.* 15.), the angle AGH will be equal to the angle GHD (*Ax.* 1.)

But when a right line interſects two other right lines, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 22.); therefore AB is parallel to CD.

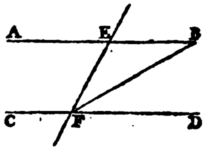
Again, ſince the angles BGH, GHD are, together, equal to two right angles (*by Hyp.*), and AGH, BGH are, alſo, equal to two right angles, (*Prop.* 13.) the angles AGH, BGH will be equal to the angles BGH, GHD (*Ax.* 1.)

And, if the common angle BGH be taken away, the remaining angle AGH will be equal to the remaining angle GHD (*Ax.* 3.)

But theſe are alternate angles: therefore, in this caſe, AB will, alſo, be parallel to CD (*Prop.* 22.) Q. E. D.

PROP. XXIV. THEOREM.

If a right line interſect two parallel right lines it will make the alternate angles equal to each other.



Let the right line EF interſect the two parallel right lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

For if they be not equal, one of them muſt be greater than the other; let EFD be the greater; and make the angle EFB equal to AEF (*Prop.* 20.)

Then, ſince AB, CD are parallel, the right line FB, which interſects CD, being produced, will meet AB in ſome point B (*Poſ.* 4.)

And, ſince EFB is a triangle, the outward angle AEF will be greater than the inward oppoſite angle EFB. (*Prop.* 16.)

But the angles AEF, EFB are equal to each other (*by Conſt.*) whence they are equal and unequal, at the ſame time, which is abſurd.

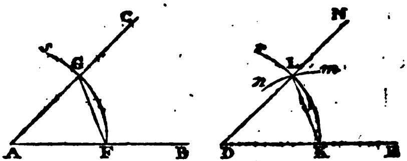
The angle EFD, therefore, is not greater than the angle AEF; and, in the ſame manner, it may be ſhewn that it is not leſs; conſequently they muſt be equal to each other. Q. E. D.

COROLL. Right lines which are perpendicular to one of two parallel right lines, are alſo perpendicular to the other.

PROP.

PROP. XX. PROBLEM.

At a given point, in a given right line, to make a rectilineal angle equal to a given rectilineal angle.



Let DE be the given right line, D the given point, and BAC the given rectilineal angle; it is required to make an angle at the point D that ſhall be equal to BAC.

Take any point F in AB, and from the point A, at the diſtance AF, deſcribe the circle FGr, cutting AC in G; and join FG.

Make DK equal to AF, and KE equal to FG (*Prop.* 3.); and from the points D, K, at the diſtances DK, KE, deſcribe the circles KLr and nLm, cutting each other in L.

Through the points D, L draw the right line DN, and the angle EDN will be equal to BAC, as was required.

For, join KL: then ſince AG is equal to AF (*Def.* 13.), and AF is equal to DK (*by Conſt.*), AG will alſo be equal to DK (*Ax.* 1.)

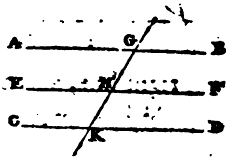
But DK is equal to DL (*Def.* 13.); conſequently AG is alſo equal to DL (*Ax.* 1.); and FG is equal to KE or KL (*by Conſt.*)

The three ſides of the triangle DKL are, therefore, equal to the three ſides of the triangle AFG, each to each; whence

COROLL. If a right line interſect two other right lines, and make the two inward angles, on the ſame ſide, together leſs than two right angles, thoſe lines, being produced, will meet each other.

PROP. XXVI. THEOREM.

Right lines which are parallel to the ſame right line, are parallel to each other.



Let the right lines AB, CD be each of them parallel to EF, then will AB be parallel to CD.

For, draw any right line GK, cutting the lines AB, EF, CD, in the points G, H and K.

Then, becauſe AB is parallel to EF (*by Hyp.*), and GH interſects them, the angle AGH is equal to the alternate angle GHF (*Prop.* 24.)

And becauſe CD is parallel to EF (*by Hyp.*), and HK interſects them, the outward angle GHF is equal to the inward angle HKD (*Prop.* 25.)

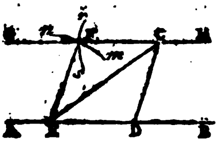
But the angle AGH has been ſhewn to be equal to the angle GHF; therefore the angle AGK is alſo equal to the angle GKD.

And, ſince the right line GK interſects the two right lines AB, CD, and makes the angle AGK equal to the alternate angle GKD; AB will be parallel to CD, as was to be ſhewn.

PROP.

PROP. XXVII. PROBLEM.

Through a given point, to draw a right line parallel to a given right line.



Let AB be the given right line, and C the given point; it is required to draw a right line through the point C that ſhall be parallel to AB.

Take any point D in AB, and make DE equal to DC (*Prop.* 3.); and from the points C, E, with the diſtances CD, ED, deſcribe the arcs *rs*, *nm*.

Then, ſince any two ſides of the triangle ECD are, together, greater than the third ſide (*Prop.* 18.), thoſe arcs will interſect each other (*Prop.* 19.)

Let them interſect at F; and through the points F, C, draw the line GH, and it will be parallel to AB, as was required.

For, ſince the ſides CF, FE of the triangle EFC are each equal to the ſide CD, or DE, of the triangle CDE, (*by Conſt.*) and EC is common, the angle ECF will be equal to the angle CED (*Prop.* 7.)

But theſe are alternate angles; therefore GH is parallel to AB (*Prop.* 24.); and it is drawn through the point C, as was to be done.

angle CBE will be equal to the remaining angle DAF (*Prop.* 29. *Cor.* 1.)

But the ſide AD is alſo equal to the ſide BC (*Prop.* 31.); conſequently, ſince the triangles ADF, BCE are mutually equiangular, and have two correſponding ſides equal to each other, they will be equal in all reſpects (*Prop.* 21.)

If, therefore, from the whole figure ABED, there be taken the triangle BCE, there will remain the parallelogram BD; and if, from the ſame figure, there be taken the triangle ADF, there will remain the parallelogram AE.

But if equal things be taken from the ſame thing, the remainders will be equal; conſequently, the parallelogram AE is equal to the parallelogram BD.

Again, let ABC, ABF be two triangles, ſtanding upon the ſame baſe AB, and between the ſame parallels, AB, CF; then will the triangle ABC be equal to the triangle ABF.

For produce CF, both ways, to D and E, and draw AD parallel to BC, and BE to AF (*Prop.* 28.)

Then, ſince BD, AE, are two parallelograms, ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE, they are equal to each other (*Prop.* 32.)

And, becauſe the diagonals AC, BF biſect them (*Prop.* 31.), the triangle ABC will alſo be equal to the triangle ABF. Q.E.D.

And if, to theſe equals, there be added the angle ABC, the angles CBD, ABC, taken together, will be equal to the three angles BCA, CAB and ABC, taken together.

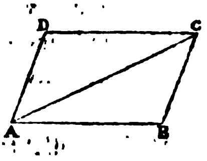
But the angles CBD, ABC, taken together, are equal to two right angles (*Prop.* 13.); conſequently the three angles BCA, CAB and ABC, taken together, are alſo equal to two right angles.

COROLL. 1. If two angles of one triangle, be equal to two angles of another, each to each, the remaining angles will alſo be equal.

COROLL. 2. Any quadrilateral may be divided into two triangles; therefore all the four angles of ſuch a figure, taken together, are equal to four right angles.

PROP. XXIX. THEOREM.

Right lines joining the correſponding extremes of two equal and parallel right lines are themſelves equal and parallel.



Let AB, DC be two equal and parallel right lines; then will the right lines AD, BC, which join the correſponding extremes of thoſe lines, be alſo equal and parallel.

For draw the diagonal, or right line AC:

Then, becauſe AB is parallel to DC, and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

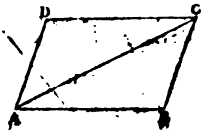
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC, and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the angle DAC will be equal to the alternate angle ACB (*Prop.* 24.)

And, becauſe the ſide DC is parallel to the ſide AB (*Def.* 22.), and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

Since, therefore, the two angles DAC, DCA, are equal to the two angles ACB, CAB, each to each, the remaining

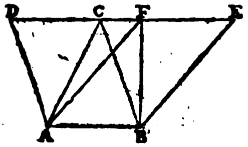
angle ADC will alſo be equal to the remaining angle ABC (*Prop.* 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* 21.)

Q.E.D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

For, ſince AD is parallel to BC (*Def.* 22.), and DE interſects them, the outward angle ECB will be equal to the inward oppoſite angle FDA. (*Prop.* 25.)

And, becauſe AF is parallel to BE (*Def.* 22.), and DE interſects them, the outward angle AFD will be equal to the inward oppoſite angle BEC (*Prop.* 25.)

Since, therefore, the angle ECB is equal to the angle FDA, and the angle AFD to the angle BEC, the remaining

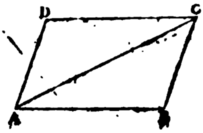
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC, and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the [illegible] will be equal to the alternate angle [illegible] *Prop.* [illegible]

And, becauſe the ſide DC is [illegible] th[illegible] (*Def.* 22.), and AC interſects the[illegible] C [illegible] be equal to the alternate angle CA[illegible]

Since, therefore, the two angles [illegible] to the two angles [illegible] CAB, each [illegible]

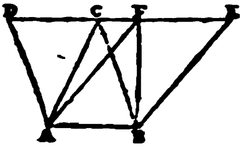
angle ADC will alſo be equal to the remaining angle ABC (*Prop.* 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* [illegible]).

Q. E. D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

[illegible] AD is parallel to BC (*Def.* 22.), and DE in-
t[illegible] outward angl[illegible] will be equal to the
in[illegible]le FDA (*P*[illegible]

[illegible] parallel [illegible] 22.), and DE
interſe[illegible]ard ang[illegible]ll be equal to
the inw[illegible] (*P*[illegible]

Since, [illegible] the angle
[illegible]nd the [illegible]maining
[illegible] angle.

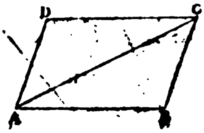
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC, and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the angle DAC will be equal to the alternate angle ACB (*Prop.* 24.)

And, becauſe the ſide DC is parallel to the ſide AB (*Def.* 22.), and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

Since, therefore, the two angles DAC, DCA, are equal to the two angles ACB, CAB, each to each, the remaining

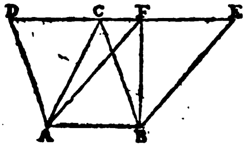
angle ADC will alſo be equal to the remaining angle ABC (*Prop.* 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* 21.)

Q.E.D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

For, ſince AD is parallel to BC (*Def.* 22.), and DE interſects them, the outward angle ECB will be equal to the inward oppoſite angle FDA (*Prop.* 25.)

And, becauſe AF is parallel to BE (*Def.* 22.), and DE interſects them, the outward angle AFD will be equal to the inward oppoſite angle BEC (*Prop.* 25.)

Since, therefore, the angle ECB is equal to the angle FDA, and the angle AFD to the angle BEC, the remaining

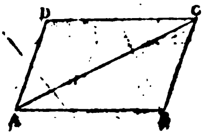
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC; and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the angle DAC will be equal to the alternate angle ACB (*Prop.* 24.)

And, becauſe the ſide DC is parallel to the ſide AB (*Def.* 22.), and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

Since, therefore, the two angles DAC, DCA, are equal to the two angles ACB, CAB, each to each, the remaining

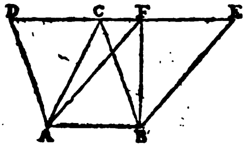
angle ADC will alſo be equal to the remaining angle ABC (*Prop.* 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* 21.)

Q.E.D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

For, ſince AD is parallel to BC (*Def.* 22.), and DE interſects them, the outward angle ECB will be equal to the inward oppoſite angle FDA. (*Prop.* 25.)

And, becauſe AF is parallel to BE (*Def.* 22.), and DE interſects them, the outward angle AFD will be equal to the inward oppoſite angle BEC (*Prop.* 25.)

Since, therefore, the angle ECB is equal to the angle FDA, and the angle AFD to the angle BEC, the remaining

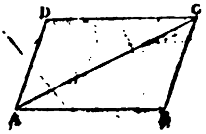
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC, and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the angle DAC will be equal to the alternate angle ACB (*Prop.* 24.)

And, becauſe the ſide DC is parallel to the ſide AB (*Def.* 22.), and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

Since, therefore, the two angles DAC, DCA, are equal to the two angles ACB, CAB, each to each, the remaining

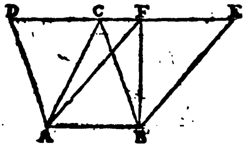
angle ADC will alſo be equal to the remaining angle ABC (*Prop*, 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* 21.)

Q.E.D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

For, ſince AD is parallel to BC (*Def.* 22.), and DE interſects them, the outward angle ECB will be equal to the inward oppoſite angle FDA (*Prop.* 25.)

And, becauſe AF is parallel to BE (*Def.* 22.), and DE interſects them, the outward angle AFD will be equal to the inward oppoſite angle BEC (*Prop.* 25.)

Since, therefore, the angle ECB is equal to the angle FDA, and the angle AFD to the angle BEC, the remaining

angle CBE will be equal to the remaining angle DAF (*Prop.* 29. *Cor.* 1.)

But the ſide AD is alſo equal to the ſide BC (*Prop.* 31.); conſequently, ſince the triangles ADF, BCE are mutually equiangular, and have two correſponding ſides equal to each other, they will be equal in all reſpects (*Prop.* 21.)

If, therefore, from the whole figure ABED, there be taken the triangle BCE, there will remain the parallelogram BD; and if, from the ſame figure, there be taken the triangle ADF, there will remain the parallelogram AE.

But if equal things be taken from the ſame thing, the remainders will be equal; conſequently, the parallelogram AE is equal to the parallelogram BD.

Again, let ABC, ABF be two triangles, ſtanding upon the ſame baſe AB, and between the ſame parallels, AB, CF; then will the triangle ABC be equal to the triangle ABF.

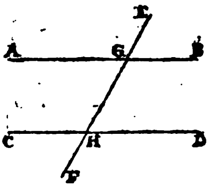
For produce CF, both ways, to D and E, and draw AD parallel to BC, and BE to AF (*Prop.* 28.)

Then, ſince BD, AE, are two parallelograms, ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE, they are equal to each other (*Prop.* 32.)

And, becauſe the diagonals AC, BF biſect them (*Prop.* 31.), the triangle ABC will alſo be equal to the triangle ABF. Q.E.D.

PROP. XXV. THEOREM.

If a right line interſect two parallel right lines, the outward angle will be equal to the inward oppoſite one, on the ſame ſide; and the two inward angles, on the ſame ſide, will be equal to two right angles.



Let the right line EF interſect the two parallel right lines AB, CD; then will the outward angle EGB be equal to the inward oppoſite angle GHD; and the two inward angles BGH, GHD will be equal to two right angles.

For, ſince the right line EF interſects the two parallel right lines AB, CD, the angle AGH will be equal to the alternate angle GHD (*Prop.* 24.)

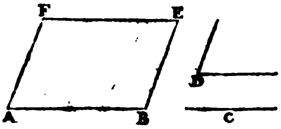
But the angle AGH is equal to the oppoſite angle EGB (*Prop.* 15.); therefore the angle EGB will, alſo, be equal to the angle GHD.

Again, ſince the right line BG falls upon the right line EF, the angles EGB, BGH, taken together, are equal to two right angles (*Prop.* 13.)

But the angle EGB has been ſhewn to be equal to the angle GHD; therefore, the angles BGH, GHD, taken together, will, alſo, be equal to two right angles. Q.E.D.

PROP. XXXIII. PROBLEM.

To make a parallelogram that ſhall have its oppoſite ſides equal to two given right lines, and one of its angles equal to a given rectilineal angle.



Let AB and C be two given right lines, and D a given rectilineal angle; it is required to make a parallelogram that ſhall have its oppoſite ſides equal to AB and C, and one of its angles equal to D.

At the point A, in the line AB, make the angle BAF equal to the angle D (*Prop.* 20.) and the ſide AF equal to C (*Prop.* 3.)

Alſo, make FE parallel and equal to AB (*Prop.* 28 *and* 3.), and join BE; then will AE be the parallelogram required.

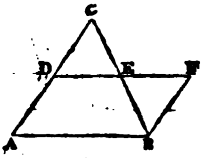
For, ſince FE is parallel and equal to AB (*by Conſt.*), BE will be parallel and equal to AF (*Prop.* 30.); whence the figure AE is a parallelogram.

And, becauſe AF is equal to C (*by Conſt.*) BE will alſo be equal to C; and the angle BAF was made equal to the angle D.

The oppoſite ſides of the parallelogram AE are, therefore, equal to the two given lines AB and C; and one of its angles is equal to the given angle D, as was to be done.

PROP. XXXIV. THEOREM,

If two ſides of a triangle be biſected, the right line joining the points of biſection, will be parallel to the baſe, and equal to one half of it.



Let ABC be a triangle, whoſe ſides CA, CB are biſected in the points D, E; then will the right line DE, joining thoſe points, be parallel to AB, and equal to one half of it.

For, in DE produced, take EF equal to ED (*Prop.* 3.), and join BF:

Then, ſince EC is equal to EB (*by Hyp.*) ED to EF (*by Conſt.*) and the angle DEC to the angle BEF (*Prop.* 15.), the ſide BF will alſo be equal to the ſide DC, or its equal DA, and the angle EFB to the angle EDC (*Prop.* 4.)

And, becauſe the right line DF interſects the two right lines CD, FB, and makes the angle EDC equal to the alternate angle EFB, BF will be parallel to DC or DA (*Prop.* 24.)

The right lines BF, AD, therefore, being equal and parallel, the lines DF, AB, joining their extremes, will alſo be equal and parallel (*Prop.* 30.)

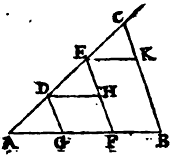
But DF is the double of DE (*by Conſt.*); conſequently AB is alſo the double of DE; that is DE is the half of AB.

Q. E. D.

PROP.

PROP. XXXV. PROBLEM.

To divide a given finite right line into any propoſed number of equal parts.



Let AB be the given right line; it is required to divide it into a certain propoſed number of equal parts.

From the point A, draw any right line AC, in which take the equal parts AD, DE, EC, at pleaſure, (*Prop.* 3.) to the number propoſed.

Join BC; and parallel thereto draw the right lines EF, DG, (*Prop.* 28.) cutting AB, in F and G; then will AB be divided into the ſame number of equal parts with AC, as was required.

For take EH, CK, each equal to DG (*Prop.* 3.), and join D, H and E, K.

Then, ſince DG is parallel to EF (*by Conſt.*), and AE interſects them, the outward angle ADG will be equal to the inward oppoſite angle DEH (*Prop.* 25.)

And, becauſe the ſides AD, DG of the triangle AGD, are equal to the ſides DE, EH of the triangle DHE (*by Conſt.*), and the angle ADG is equal to the angle DEH, the baſe AG will alſo be equal to the baſe DH, and the angle DAG to the angle EDH (*Prop.* 4.)

But, ſince the right line AE interſects the two right lines, AG, DH, and makes the outward angle EDH equal to the inward oppoſite angle DAG, DH will be parallel to AG or GF (*Prop.* 23.)

And, in the ſame manner it may be ſhown, that EK is equal to AG, and parallel to AG or FB.

The figures GH, FK, therefore, being parallelograms, the ſide DH will be equal to the ſide GF, and the ſide EK to the ſide FB (*Prop.* 31.)

But DH, EK have been each proved to be equal to AG; conſequently GF, FB are, alſo, each equal to AG; whence the line AB is divided into the ſame number of equal parts with AC, as was to be done.

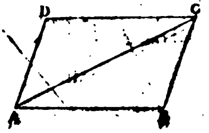
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC, and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the angle DAC will be equal to the alternate angle ACB (*Prop.* 24.)

And, becauſe the ſide DC is parallel to the ſide AB (*Def.* 22.), and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

Since, therefore, the two angles DAC, DCA, are equal to the two angles ACB, CAB, each to each, the remaining

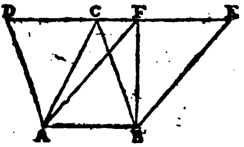
angle ADC will alſo be equal to the remaining angle ABC (*Prop.* 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* 21.)

Q.E.D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

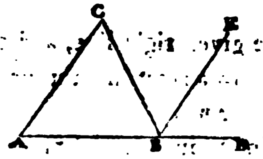
For, ſince AD is parallel to BC (*Def.* 22.), and DE interſects them, the outward angle ECB will be equal to the inward oppoſite angle FDA (*Prop.* 25.)

And, becauſe AF is parallel to BE (*Def.* 22.), and DE interſects them, the outward angle AFD will be equal to the inward oppoſite angle BEC (*Prop.* 25.)

Since, therefore, the angle ECB is equal to the angle FDA, and the angle AFD to the angle BEC, the remaining

PROP. XXVIII. THEOREM.

If one ſide of a triangle be produced, the outward angle will be equal to the two inward oppoſite angles, taken together; and the three angles of every triangle, taken together, are equal to two right angles.



Let ABC be a triangle, having one of its ſides AB produced to D; then will the outward angle CBD be equal to the two inward oppoſite angles BCA, CAB, taken together; and the three angles BCA, CAB and ABC, taken together, are equal to two right angles.

For through the point B, draw the right line BE parallel to AC (*Prop.* 28.)

Then, becauſe BE is parallel to AC, and CB interſects them, the angle CBE will be equal to the alternate angle BCA (*Prop.* 24.)

And becauſe BE is parallel to AC, and AD interſects them, the outward angle EBD will be equal to the inward angle CAB (*Prop.* 25.)

But the angles CBE, EBD are equal to the whole angle CBD; therefore the outward angle CBD is equal to the two inward oppoſite angles BCA, CAB taken together.

And if, to theſe equals, there be added the angle ABC, the angles CBD, ABC, taken together, will be equal to the three angles BCA, CAB and ABC, taken together.

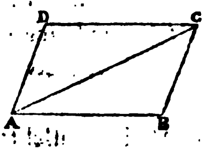
But the angles CBD, ABC, taken together, are equal to two right angles (*Prop.* 13.); conſequently the three angles BCA, CAB and ABC, taken together, are alſo equal to two right angles.

Coroll. 1. If two angles of one triangle, be equal to two angles of another, each to each, the remaining angles will alſo be equal.

Coroll. 2. Any quadrilateral may be divided into two triangles; therefore all the four angles of ſuch a figure, taken together, are equal to four right angles.

PROP. XXIX. Theorem.

Right lines joining the correſponding extremes of two equal and parallel right lines are themſelves equal and parallel.



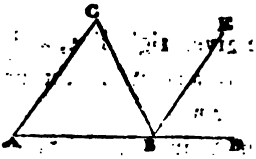
Let AB, DC be two equal and parallel right lines; then will the right lines AD, BC, which join the correſponding extremes of thoſe lines, be alſo equal and parallel.

For draw the diagonal, or right line AC:

Then, becauſe AB is parallel to DC, and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

PROP. XXVIII. THEOREM.

If one ſide of a triangle be produced, the outward angle will be equal to the two inward oppoſite angles, taken together; and the three angles of every triangle, taken together, are equal to two right angles.



Let ABC be a triangle, having one of its ſides AB produced to D; then will the outward angle CBD be equal to the two inward oppoſite angles BCA, CAB, taken together; and the three angles BCA, CAB and ABC, taken together, are equal to two right angles.

For through the point B, draw the right line BE parallel to AC (*Prop.* 28.)

Then, becauſe BE is parallel to AC, and CB interſects them, the angle CBE will be equal to the alternate angle BCA (*Prop.* 24.)

And becauſe BE is parallel to AC, and AD interſects them, the outward angle EBD will be equal to the inward angle CAB (*Prop.* 25.)

But the angles CBE, EBD are equal to the whole angle CBD; therefore the outward angle CBD is equal to the two inward oppoſite angles BCA, CAB taken together.

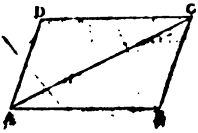
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC, and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the angle DAC will be equal to the alternate angle ACB (*Prop.* 24.)

And, becauſe the ſide DC is parallel to the ſide AB (*Def.* 22.), and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

Since, therefore, the two angles DAC, DCA, are equal to the two angles ACB, CAB, each to each, the remaining

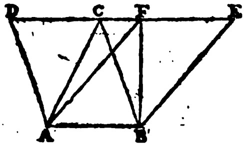
angle ADC will alſo be equal to the remaining angle ABC (*Prop*, 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* 21.)

Q.E.D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

For, ſince AD is parallel to BC (*Def.* 22.), and DE interſects them, the outward angle ECB will be equal to the inward oppoſite angle FDA (*Prop.* 25.)

And, becauſe AF is parallel to BE (*Def.* 22.), and DE interſects them, the outward angle AFD will be equal to the inward oppoſite angle BEC (*Prop.* 25.)

Since, therefore, the angle ECB is equal to the angle FDA, and the angle AFD to the angle BEC, the remaining

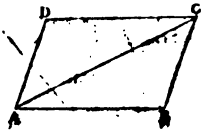
And, becauſe AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC, ADC, and the angle DCA equal to the angle CAB, the ſide AD will alſo be equal to the ſide BC, and the angle DAC to the angle ACB (*Prop.* 4.)

Since, therefore, the right line AC interſects the two right lines AD, BC, and makes the alternate angles equal to each other, thoſe lines will be parallel (*Prop.* 23.)

But the line AD has been proved to be equal to the line BC; conſequently they are both equal and parallel. Q.E.D.

PROP. XXX. THEOREM.

The oppoſite ſides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let ABCD be a parallelogram, whoſe diagonal is AC; then will its oppoſite ſides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, ſince the ſide AD is parallel to the ſide BC (*Def.* 22.), and the right line AC interſects them, the angle DAC will be equal to the alternate angle ACB (*Prop.* 24.)

And, becauſe the ſide DC is parallel to the ſide AB (*Def.* 22.), and AC interſects them, the angle DCA will be equal to the alternate angle CAB (*Prop.* 24.)

Since, therefore, the two angles DAC, DCA, are equal to the two angles ACB, CAB, each to each, the remaining

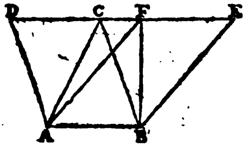
angle ADC will alſo be equal to the remaining angle ABC (*Prop.* 29. *Cor.*) and the whole angle DAB to the whole angle DCB.

But, the triangles CDA, ABC, being mutually equiangular, and having AC common, the ſide DC will alſo be equal to the ſide AB, and the ſide AD to the ſide BC, and the two triangles will be equal in all reſpects (*Prop* 21.)

Q.E.D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, ſtanding upon the ſame baſe, and between the ſame parallels, are equal to each other.



Let AE, BD be two parallelograms ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AE be equal to the parallelogram BD.

For, ſince AD is parallel to BC (*Def.* 22.), and DE interſects them, the outward angle ECB will be equal to the inward oppoſite angle FDA (*Prop.* 25.)

And, becauſe AF is parallel to BE (*Def.* 22.), and DE interſects them, the outward angle AFD will be equal to the inward oppoſite angle BEC (*Prop.* 25.)

Since, therefore, the angle ECB is equal to the angle FDA, and the angle AFD to the angle BEC, the remaining

angle CBE will be equal to the remaining angle DAF (*Prop.* 29. *Cor.* 1.)

But the ſide AD is alſo equal to the ſide BC (*Prop.* 31.); conſequently, ſince the triangles ADF, BCE are mutually equiangular, and have two correſponding ſides equal to each other, they will be equal in all reſpects (*Prop.* 21.)

If, therefore, from the whole figure ABED, there be taken the triangle BCE, there will remain the parallelogram BD; and if, from the ſame figure, there be taken the triangle ADF, there will remain the parallelogram AE.

But if equal things be taken from the ſame thing, the remainders will be equal; conſequently, the parallelogram AE is equal to the parallelogram BD.

Again, let ABC, ABF be two triangles, ſtanding upon the ſame baſe AB, and between the ſame parallels, AB, CF; then will the triangle ABC be equal to the triangle ABF.

For produce CF, both ways, to D and E, and draw AD parallel to BC, and BE to AF (*Prop.* 28.)

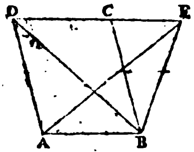
Then, ſince BD, AE, are two parallelograms, ſtanding upon the ſame baſe AB, and between the ſame parallels AB, DE, they are equal to each other (*Prop.* 32.)

And, becauſe the diagonals AC, BF biſect them (*Prop.* 31.), the triangle ABC will alſo be equal to the triangle ABF. Q.E.D.

PROP.

PROP. XXXII. THEOREM.

If a parallelogram and a triangle ſtand upon the ſame baſe, and between the ſame parallels, the parallelogram will be double the triangle.



Let the parallelogram AC and the triangle AEB ſtand upon the ſame baſe AB, and between the ſame parallels AB, DE; then will the parallelogram AC be double the triangle AEB.

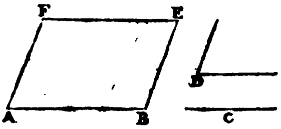
For join the points B, D; then will the parallelogram AC be double the triangle ADB, becauſe the diagonal DB divides it into two equal parts (*Prop.* 31.)

But the triangle ADB is equal to the triangle AEB, becauſe they ſtand upon the ſame baſe AB, and between the ſame parallels AB, DE (*Prop.* 32.); whence the parallelogram AC is alſo double the triangle AEB. Q.E.D.

COROLL. If the baſe of the parallelogram be half that of the triangle, or the baſe of the triangle be double that of the parallelogram, the two figures will be equal to each other.

PROP. XXXIII. PROBLEM.

To make a parallelogram that ſhall have its oppoſite ſides equal to two given right lines, and one of its angles equal to a given rectilineal angle.



Let AB and C be two given right lines, and D a given rectilineal angle; it is required to make a parallelogram that ſhall have its oppoſite ſides equal to AB and C, and one of its angles equal to D.

At the point A, in the line AB, make the angle BAF equal to the angle D (*Prop.* 20.) and the ſide AF equal to C (*Prop.* 3.)

Alſo, make FE parallel and equal to AB (*Prop.* 28 *and* 3.), and join BE; then will AE be the parallelogram required.

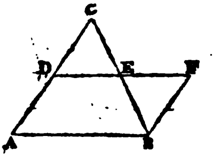
For, ſince FE is parallel and equal to AB (*by Conſt.*), BE will be parallel and equal to AF (*Prop.* 30.); whence the figure AE is a parallelogram.

And, becauſe AF is equal to C (*by Conſt.*) BE will alſo be equal to C; and the angle BAF was made equal to the angle D.

The oppoſite ſides of the parallelogram AE are, therefore, equal to the two given lines AB and C; and one of its angles is equal to the given angle D, as was to be done.

PROP. XXXIV. THEOREM.

If two ſides of a triangle be biſected, the right line joining the points of biſection, will be parallel to the baſe, and equal to one half of it.



Let ABC be a triangle, whoſe ſides CA, CB are biſected in the points D, E; then will the right line DE, joining thoſe points, be parallel to AB, and equal to one half of it.

For, in DE produced, take EF equal to ED (*Prop.* 3.), and join BF:

Then, ſince EC is equal to EB (*by Hyp.*) ED to EF (*by Conſt.*) and the angle DEC to the angle BEF (*Prop.* 15.), the ſide BF will alſo be equal to the ſide DC, or its equal DA, and the angle EFB to the angle EDC (*Prop.* 4.)

And, becauſe the right line DF interſects the two right lines CD, FB, and makes the angle EDC equal to the alternate angle EFB, BF will be parallel to DC or DA (*Prop.* 24.)

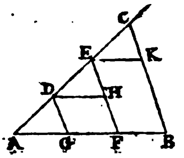
The right lines BF, AD, therefore, being equal and parallel, the lines DF, AB, joining their extremes, will alſo be equal and parallel (*Prop.* 30.)

But DF is the double of DE (*by Conſt.*); conſequently AB is alſo the double of DE; that is DE is the half of AB.

Q.E.D.

PROP. XXXV. PROBLEM.

To divide a given finite right line into any propoſed number of equal parts.



Let AB be the given right line; it is required to divide it into a certain propoſed number of equal parts.

From the point A, draw any right line AC, in which take the equal parts AD, DE, EC, at pleaſure, (*Prop.* 3.) to the number propoſed.

Join BC; and parallel thereto draw the right lines EF, DG, (*Prop.* 28.) cutting AB, in F and G; then will AB be divided into the ſame number of equal parts with AC, as was required.

For take EH, CK, each equal to DG (*Prop.* 3.), and join D, H and E, K.

Then, ſince DG is parallel to EF (*by Conſt.*), and AE interſects them, the outward angle ADG will be equal to the inward oppoſite angle DEH (*Prop.* 25.)

And, becauſe the ſides AD, DG of the triangle AGD, are equal to the ſides DE, EH of the triangle DHE (*by Conſt.*), and the angle ADG is equal to the angle DEH, the baſe AG will alſo be equal to the baſe DH, and the angle DAG to the angle EDH (*Prop.* 4.)

But, ſince the right line AE interſects the two right lines, AG, DH, and makes the outward angle EDH equal to the inward oppoſite angle DAG, DH will be parallel to AG or GF (*Prop.* 23.)

And, in the ſame manner it may be ſhown, that EK is equal to AG, and parallel to AG or FB.

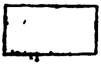
The figures GH, FK, therefore, being parallelograms, the ſide DH will be equal to the ſide GF, and the ſide EK to the ſide FB (*Prop.* 31.)

But DH, EK have been each proved to be equal to AG; conſequently GF, FB are, alſo, each equal to AG; whence the line AB is divided into the ſame number of equal parts with AC, as was to be done.

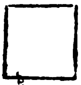
BOOK II.

DEFINITIONS.

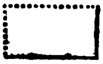
1. A rectangle is a parallelogram whoſe angles are all right angles.



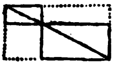
2. A ſquare is a rectangle, whoſe ſides are all equal to each other.



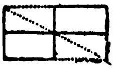
3. Every rectangle is ſaid to be contained by any two of the right lines which contain one of the right angles.



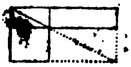
4. If two right lines be drawn through any point in the diagonal of a parallelogram, parallel to its oppoſite ſides, the figures which are interſected by the diagonal are called parallelograms about the diagonal.



5. And the other two parallelograms, which are not interſected by the diagonal, are called complements to the parallelograms which are about the diagonal.



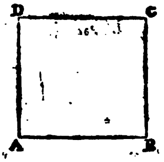
6. In every parallelogram, either of the two parallelograms about the diagonal, together with the two complements, is called a gnomon.



7. The altitude of any figure is a perpendicular drawn from the vertical angle to the baſe.

PROP. I. PROBLEM.

Upon a given right line to deſcribe a ſquare.



Let AB be the given right line; it is required to deſcribe a ſquare upon it.

Make AD, BC, each perpendicular and equal to AB (I. 11 *and* 3.), and join DC; then will AC be the ſquare required.

For, ſince the angles DAB, ABC are right angles (*by Conſt.*), AD will be parallel to BC (I. 22 *Cor.*)

And becauſe AD, BC are equal and parallel, AB, DC will, alſo, be equal and parallel (I. 30.)

But AD, BC are each equal to AB (*by Conſt.*); whence AD, AB, BC and CD are all equal to each other.

The figure AC, therefore, is an equilateral parallelogram; and it has, likewiſe, all its angles right angles.

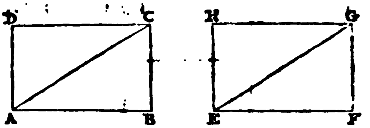
For the angle DAB is equal to the angle BCB, and the angle ABC to the angle ADC (I. 30.)

But the angles DAB, ABC are right angles (*by Conſt.*); conſequently the angles DCB, ADC are, alſo, right angles.

The figure AC, therefore, being both equilateral and rectangular, is a ſquare; and it is deſcribed upon the line AB, as was to be done.

PROP. II. THEOREM.

Rectangles and Squares contained under equal lines are equal to each other.



Let BD, FH be two rectangles, having the ſides AB, BC equal to the ſides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH.

For draw the diagonals AC, EG:

Then, ſince the two ſides AB, BC are equal to the two ſides EF, FG, each to each (*by Hyp.*), and the angle B is equal to the angle F (I. 8.), the triangle ABC will be equal to the triangle EFG (I. 4.)

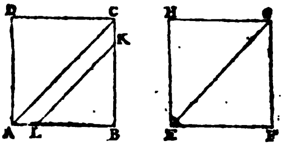
But the diagonal of every parallelogram divides it into two equal parts (I. 30.); whence the halves being equal, the wholes will alſo be equal.

The rectangle BD is, therefore, equal to the rectangle FH; and in the ſame manner it may be proved when the figures are ſquares. Q. E. D.

PROP.

PROP. III. THEOREM.

The ſides and diagonals of equal ſquares are equal to each other.



Let BD, FH, be two equal ſquares; then will the ſide AB be equal to the ſide EF, and the diagonal AC to the diagonal EG.

For if AB, EF be not equal, one of them muſt be greater than the other; let AB be the greater, and make BL, BK each equal to EF or FG (I. 3.); and join LK.

Then, becauſe BL is equal to FE, BK to FG, and the angle LBK to the angle EFG, being each of them right angles, the triangle BLK will be equal to the triangle FEG (I. 4.)

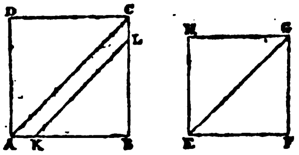
But the triangle FEG is equal to the triangle BAC, being each of them the halves of the equal ſquares FH, BD (I. 30.); whence the triangle BLK is alſo equal to the triangle BAC, the leſs to the greater, which is abſurd.

The ſide AB, therefore is not greater than the ſide EF; and in the ſame manner it may be proved that it cannot be leſs; conſequently they are equal to each other.

And becauſe AB is equal to EF, BC to FG, and the angle ABC to the angle EFG (I. 8.), the ſide AC will alſo be equal to the ſide EG (I. 4.) Q. E. D.

PROP. IV. THEOREM.

The ſquare of a greater line is greater than the ſquare of a leſs; and the greater ſquare has the greater ſide.



Let the right line AB be greater than the right line EF; then will BD, the ſquare of AB, be greater than FH, the ſquare of EF.

For ſince AB is greater than EF, and BC than FG (*by Hyp.*), take BK, a part of BA, equal to EF, and BL, a part of BC, equal to FG (I. 3.); and join KL.

Then, becauſe BK is equal to FE, BL to FG, and the angle KBL to the angle EFG (I. 8.), the triangle BLK will be equal to the triangle FGE (I. 4.)

But the triangle BCA is greater than the triangle BLK, whence it is alſo greater than the triangle FGE.

And ſince the ſquare BD is double the triangle BCA, and the ſquare FH is double the triangle FGE (I. 30.), the ſquare BD will alſo be greater than the ſquare FH.

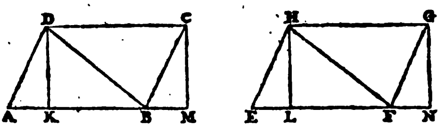
Again, let the ſquare BD be greater than the ſquare FH; then will the ſide AB be greater than the ſide EF.

For if AB be not greater than EF, it muſt be either equal to it, or leſs; but it cannot be equal to it, for then the ſquare BD would be equal to the ſquare FH (II. 2.), which it is not.

Neither can it be leſs, for then the ſquare BD would be leſs than the ſquare FH (II. 4.), which it is not; conſequently AB is greater than EE, as was to be ſhewn.

PROP. V. THEOREM.

Parallelograms or triangles, having equal baſes and altitudes, are equal to each other.



Let AC, EG be two parallelograms, having the baſe AB equal to the baſe EF, and the altitude DK to the altitude HL; then will the parallelogram AC be equal to the parallelogram EG.

For upon AB, EF, produced if neceſſary, let fall the perpendiculars CM, GN (I. 12.)

Then, ſince MD, NH are rectangular parallelograms, the ſide DC is equal to the ſide KM, and the ſide HG to the ſide LN (I. 30.)

But DC is alſo equal to AB, and HG to EF (I. 30.); therefore KM is equal to AB, and LN to EF.

And, ſince AB is equal to EF (*by Hyp.*), KM will be equal to LN; and conſequently the rectangle MD is equal to the rectangle NH (II. 2.)

But the rectangle MD is equal to the parallelogram AC, becauſe they ſtand upon the ſame baſe DC, and between the ſame parallels DC, AM.

And, for the ſame reaſon, the rectangle NH is equal to the parallelogram EG; whence the parallelogram AC is equal to the parallelogram EG.

Again, let ABD, EFH be two triangles, having the baſe AB equal to the baſe EF, and the altitude DK to the altitude HL; then will the triangle ABD be equal to the triangle EFH.

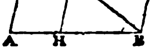
For, if the parallelograms AC, EG be compleated, they will be equal to each other, by the former part of the propoſition.

And ſince the diagonals, DB, HF divide them into two equal parts (I. 30.) the triangle ABD will alſo be equal to the triangle EFH. Q. E. D.

COROLL. Parallelograms or Triangles ſtanding upon equal baſes, and between the ſame parallels, are equal to each other.

PROP. VI. THEOREM.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to each other.



Let AC be a parallelogram, and AK, KC, complements about the diagonal BD; then will the complement AK be equal to the complement KC.

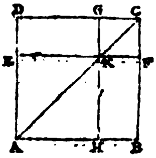
For ſince AC is a parallelogram, whoſe diagonal is BD, the triangle DAB will be equal to the triangle BCD (I. 30.)

And, becauſe EG, HF are alſo parallelograms, whoſe diagonals are DK, KB, the triangle DGK will be equal to the triangle DEK, and the triangle KFB to the triangle KHB (I. 30.)

But, ſince the triangles DGK, KFB are together, equal to the triangles DEK, KHB, and the whole triangle DAB to the whole triangle DCB, the remaining part AK will be equal to the remaining part KC. Q. E. D.

PROP. VII. THEOREM.

Parallelograms which are about the diagonal of a ſquare are themſelves ſquares.



Let BD be a ſquare, and HE, FG parallelograms about its diagonal AC; then will thoſe parallelograms alſo be ſquares.

For ſince the ſide of the ſquare AB is equal to the ſide BC, the angle CAB will be equal to the angle ACB (I. 5.)

And becauſe the right line GH is parallel to the right line CB, the angle AKH will alſo be equal to the angle ACB (I. 25.)

The angles CAB, AKH are, therefore, equal to each other; and conſequently the ſide AH is equal to the ſide HK (I. 6.)

But the ſide AH is equal to the ſide EK, and the ſide HK to the ſide AE (I. 30.); whence the figure HE is equilateral.

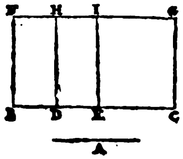
It has alſo all its angles right angles:

For EAH is a right angle, being the angle of a ſquare; and HG, EF are each of them parallel to the ſides of the ſame ſquare, whence the remaining angles will alſo be right angles (I. 25.)

The figure HE, therefore, being equilateral, and having all its angles right angles, is a ſquare; and the ſame may be proved of the figure FG. Q. E. D.

PROP. VIII. THEOREM.

The rectangles contained under a given line and the ſeveral parts of another line, any how divided, are, together, equal to the rectangle of the two whole lines.



Let A and BC be two right lines, one of which, BC, is divided into ſeveral parts in the points D, E; then will the rectangle of A and BC, be equal to the ſum of the rectangles of A and BD, A, and DE, and A and EC.

For make BF perpendicular to BC (I. 11.) and equal to A (I. 3.), and draw FG parallel to BC, and DH, EI and CG each parallel to BF (I. 27.) producing them till they meet FG in the points H, I, G.

Then, ſince the rectangle BH is contained by BD and BF (II. *Def.* 3.), it is alſo contained by BD and A, becauſe BF is equal to A (*by Conſt.*)

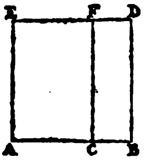
And, ſince the rectangle DI is contained by DE and DH, it is alſo contained by DE and A, becauſe DH is equal to BF (I. 30.), or A.

The rectangle EG, in like manner, is contained by EC and [illegible] and the rectangle BG by BC and A.

But the whole rectangle BG, is equal to the rectangles BH, DI and EG, taken together; whence the rectangle of A and BC is alſo equal to the rectangles of A and BD, A and DE and A and EC, taken together. Q. E. D.

PROP. IX. THEOREM.

If a right line be divided into any two parts, the rectangles of the whole line and each of the parts, are, together, equal to the ſquare of the whole line.



Let the right line AB be divided into any two parts in the point C; then will the rectangle of AB, AC, together with the rectangle of AB, BC, be equal to the ſquare of AB.

For, upon AB deſcribe the ſquare AD (II. 1.), and through C draw CF parallel to AE or BD (I. 27.)

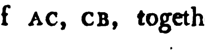
Then, ſince the rectangle AF is contained by AE, AC, it is alſo contained by AB, AC, becauſe AE is equal to AB (II. *Def.* 2.)

And, ſince the rectangle CD is contained by BD, BC, it is alſo contained by AB, BC, becauſe BD is equal to AB.

But AD, or the ſquare of AB, is equal to the rectangles AF, CD, taken together; whence the rectangle of AB, AC, together with the rectangle of AB, BC, is alſo equal to the ſquare of AB. Q. E. D.

PROP. X. THEOREM.

If a right line be divided into any two parts, the rectangle of the whole line and one of the parts, is equal to the rectangle of the two parts, together with the ſquare of the aforeſaid part.



Let the right line AB be divided into any two parts in the point C; then will the rectangle of AB, BC be equal to the rectangle of AC, CB, together with the ſquare of CB.

For upon CB deſcribe the ſquare CE (II. 1.), and through A draw AF parallel to CD (I. 27.) meeting ED, produced, in F.

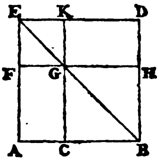
Then, ſince AE is a rectangle, contained by AB, BE, it is alſo contained by AB, BC, becauſe BE is equal to BC (II. *Def.* 2.)

And, in like manner, AD is a rectangle contained by AC, CD, or by AC, CB; and CE is the ſquare of CB (*by Conſt.*)

But the rectangle AE is equal to the rectangle AD, and the ſquare CE, taken together; whence the rectangle of AB, BC is alſo equal to the rectangle of AC, CB together with the ſquare of CB. Q. E. D.

PROP. XI. THEOREM.

If a right line be divided into any two parts, the ſquare of the whole line will be equal to the ſquares of the two parts, together with twice the rectangle of thoſe parts.



Let the right line AB be divided into any two parts in the point C; then will the ſquare of AB be equal to the ſquares of AC, CB together with twice the rectangle of AC, CB.

For upon AB make the ſquare AD (II. 1.), and draw the diagonal EB; and make CK, FH parallel to AE, ED (I. 27.)

Then, ſince the parallelograms about the diagonal of a ſquare are themſelves ſquares (II. 7.), FK will be the ſquare of FG, or its equal AC, and CH of CB.

And ſince the complements of the parallelograms about the diagonal are equal to each other (II. 6.), the complement AG will be equal to the complement GD.

But AG is equal to the rectangle of AC, CB, becauſe CG is equal to CB (II. *Def.* 2.); and GD is alſo equal to the rectangle of AC, CB, becauſe GK is equal to GF (*Def.* II. 2.) or AC (I. 30.), and GH to CB (I. 30.)

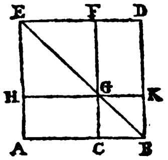
The two rectangles AG, GD are, therefore, equal to twice the rectangle of AC, CB; and FK, CH have been proved to be equal to the ſquares of AC, CB.

But theſe two rectangles, together with the two ſquares, make up the whole ſquare AD; conſequently the ſquare AD is equal to the ſquares of AC, CB, together with twice the rectangle of AC, CB. Q. E. D.

COROLL. If a line be divided into two equal parts, the ſquare of the whole line will be equal to four times the ſquare of half the line.

PROP. XII. THEOREM.

If a right line be divided into any two parts, the ſquares of the whole line, and one of the parts, are equal to twice the rectangle of the whole line and that part, together with the ſquare of the other part.



Let the right line AB be divided into any two parts in the point C; then will the ſquares of AB, BC, be equal to twice the rectangle of AB, BC together with the ſquare of AC.

For, upon AB make the ſquare AD (II. 1.), and draw the diagonal BE; and make FC, HK parallel to BD, BA (I. 27.):

Then becauſe AG is equal to GD (II. 6.), to each of theſe equals add CK, and the whole AK will be equal to the whole CD.

And, ſince the doubles of equals are equal, the gnomon HBF, together with CK, will be the double of AK.

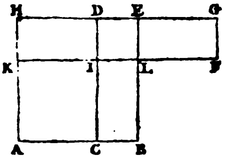
But CK is a ſquare upon CB (II. 7.), and twice the rectangle of AB, BC is the double of AK, whence the gnomon HBF, together with the ſquare CK, is, alſo, equal to twice the rectangle AB, BC.

And, becauſe HF is a ſquare upon HG or AC (II. 7.), if this be added to each of theſe equals, the gnomon HBF, together with the ſquares CK, HF, will be equal to twice the rectangle AB, BC, together with the ſquare of AC.

But the gnomon HBF, together with the ſquares CK, HF, are equal to the whole ſquare AD, together with the ſquare CK; conſequently, the ſquares of AB, BC, are equal to twice the rectangle of AB, BC together with the ſquare of AC. Q. E. D.

PROP. XIII. THEOREM.

The difference of the ſquares of any two unequal lines, is equal to a rectangle under their ſum and difference.



Let AB, AC be any two unequal lines; then will the difference of the ſquares of thoſe lines be equal to a rectangle under their ſum and difference.

For upon AB, AC make the ſquares AE, AI (II. 1.); and in HE, produced, take EG equal to AC (I. 3.); and make GF parallel to EB (I. 27.); and produce CI, IK till they meet HG, GF in D and F.

Then, ſince HE is equal to AB (*Def.* II. 2.) and EG to AC (*by Conſt.*), HG will be equal to the ſum of AB and AC.

And becauſe AH is equal to AB, and AK to AC (II. *Def.* 2.), KH will be equal to CB, or the difference of AB and AC.

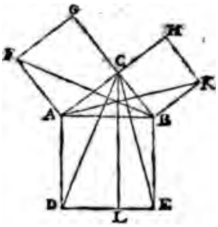
But the rectangle KG is contained by HG and HK, whence it is, alſo, contained by the ſum and difference of AB and AC.

And, ſince LE is equal to HK (I. 30.) or CB (*by Conſt.*), and EG to AC (*by Conſt.*) CI, or LB, the rectangle IG will be equal to the rectangle LC (II. 2.)

But the rectangles HL, LC are, together, equal to the difference of the ſquares AE, AI; conſequently the rectangles HL, LG, or the whole rectangle KG, is alſo equal to the difference of thoſe ſquares. Q. E. D.

PROP. XIV. THEOREM.

In any right angled triangle, the ſquare of the hypotenuſe is equal to the ſum of the ſquares of the other two ſides.



Let ABC be a right angled triangle, having the right angle ACB; then will the ſquare of the hypotenuſe AB be equal to the ſum of the ſquares of AC and CB.

For, on AB, deſcribe the ſquare AE (II. 1.), and on AC, CB the ſquares AG, BH; and through the point C, draw CL parallel to AD or BE (I. 27.) and join BF, CD, AK and CE.

Then, ſince the right line AC meets the two right lines GC, CB in the point C, and makes each of the angles ACG, ACB a right angle (*by Hyp. and Def.* 2.), GC will be in the ſame right line with CB (I. 14.)

And, becauſe the angle FAC is equal to the angle DAB (I. 8.), if the angle CAB be added to each of them, the whole angle FAB will be equal to the whole angle DAC.

The ſides FA, AB, are, alſo, equal to the ſides CA, AD, each to each, (*Def.* 2.), and their included angles have, likewiſe, been ſhewn to be equal; whence the triangle ABF is equal to the triangle ACD (I. 4.)

But the ſquare AG is double the triangle ABF (I. 32.) and the parallelogram AL is double the triangle ACD (I. 32.); conſequently the parallelogram AL is equal to the ſquare AG (*Ax.* 6.)

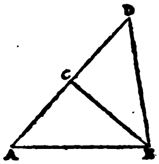
And, in the ſame manner, it may be demonſtrated, that the parallelogram BL is equal to the ſquare BH; therefore the whole ſquare AE is equal to the ſquares AG and BH taken together. Q. E. D.

COROLL. The difference of the ſquares of the hypotenuſe and either of the other ſides is equal to the ſquare of the remaining ſide.

PROP.

PROP. XV. THEOREM.

If the ſquare of one of the ſides of a triangle be equal to the ſum of the ſquares of the other two ſides, the angle contained by thoſe ſides will be a right angle.



Let ABC be a triangle; then if the ſquare of the ſide AB be equal to the ſum of the ſquares of AC, CB, the angle ACB will be a right angle.

For, at the point C, make CD at right angles to CB (I. 11.), and equal to AC (I. 3.); and join DB.

Then, ſince the ſquares of equal lines are equal (II. 2.), the ſquare of DC will be equal to the ſquare of AC.

And, if, to each of theſe equals, there be added the ſquare of CB, the ſquares of DC, CB will be equal to the ſquares of AC, CB.

But the ſquares of DC, CB are equal to the ſquare of BD (II. 14.), and the ſquares of AC, CB to the ſquare of AB (*by Hyp.*); whence the ſquare of BD is equal to the ſquare of AB.

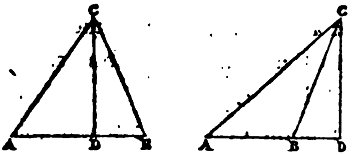
And ſince equal ſquares have equal ſides (II. 3.), AB is equal to BD; BC is alſo common to each of the triangles ABC, DBC, and AC is equal to CD (*by Conſt.*);

conſequently the angle ACB is equal to the angle BCD (I. 7.)

But the angle BCD is a right angle (*by Conſt.*), whence the angle ACB is alſo a right angle. Q. E. D.

PROP. XVI. THEOREM.

The difference of the ſquares of the two ſides of any triangle, is equal to the difference of the ſquares of the two lines, or diſtances, included between the extremes of the baſe and the perpendicular.



Let ABC be a triangle, having CD perpendicular to AB; then will the difference of the ſquares of AC, CB be equal to the difference of the ſquares of AD, DB.

For the ſum of the ſquares of AD, DC is equal to the ſquare of AC (II. 14.); and the ſum of the ſquares of BD, DC is equal to the ſquare of BC (II. 14.)

The difference, therefore, between the ſum of the ſquares of AD, DC, and the ſum of the ſquares of BD, DC is equal to the difference of the ſquares of AC, CB.

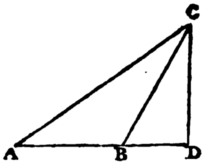
And, ſince DC is common, the difference between the ſum of the ſquares of AD, DC, and the ſum of the ſquares of BD, DC is equal to the difference of the ſquares of AD, DB.

But things which are equal to the ſame thing are equal to each other; conſequently the difference of the ſquares of AC, CB is equal to the difference of the ſquares of AD, DB. Q.E.D.

COROLL. The rectangle under the ſum and difference of the two ſides of any triangle, is equal to the rectangle under the baſe and the difference of the ſegments of the baſe (II. 18.)

PROP. XVII. THEOREM.

In any obtuſe-angled triangle, the ſquare of the ſide ſubtending the obtuſe angle, is greater than the ſum of the ſquares of the other two ſides, by twice the rectangle of the baſe and the diſtance of the perpendicular from the obtuſe angle.



Let ABC be a triangle, of which ABC is an obtuſe angle, and CD perpendicular to AB; then will the ſquare of AC be greater than the ſquares of AB, BC, by twice the rectangle of AB, BD.

For, ſince the right line AD is divided into two parts, in the point B, the ſquare of AD is equal to the ſquares of

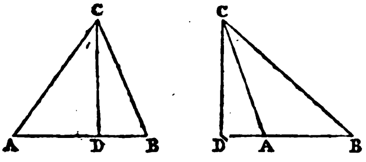
AB, BD, together with twice the rectangle of AB, BD (II. 11.)

And if, to each of theſe equals, there be added the ſquare of DC, the ſquares of AD, DC will be equal to the ſquares of AB, BD and DC, together with twice the rectangle of AB, BD.

But the ſquares of AD, DC are equal to the ſquare of AC, and the ſquares of BD, DC to the ſquare of BC (II. 14.); whence the ſquare of AC is greater than the ſquares of AB, BC by twice the rectangle of AB, BD. Q.E.D.

PROP. XVIII. THEOREM.

In any triangle, the ſquare of the ſide ſubtending an acute angle, is leſs than the ſum of the ſquares of the baſe and the other ſide, by twice the rectangle of the baſe and the diſtance of the perpendicular from the acute angle.



Let ABC be a triangle, of which ABC is an acute angle, and CD perpendicular to AB: then will the ſquare of AC be leſs than the ſum of the ſquares of AB and BC, by twice the rectangle of AB, BD.

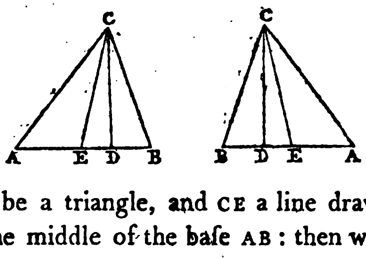
For, ſince AB, or AB produced, is divided into two parts in the points D, or A, the ſum of the ſquares of AB, BD is equal to twice the rectangle of AB, BD, together with the ſquare of AD (II. 12.)

And if, to each of theſe equals, there be added the ſquare of DC, the ſum of the ſquares of AB, BD and DC will be equal to twice the rectangle of AB, BD, together with the ſum of the ſquares of AD, DC.

But the ſum of the ſquares of BD, DC is equal to the ſquare of BC, and the ſum of the ſquares of AD, DC to the ſquare of AC (II. 14.); whence the ſquare of AC is leſs than the ſum of the ſquares of AB, BC, by twice the rectangle of AB, BD. Q.E.D.

PROP. XIX. THEOREM.

In any triangle, the double of the ſquare of a line drawn from the vertex to the middle of the baſe, together with double the ſquare of the ſemi-baſe, is equal to the ſum of the ſquares of the other two ſides.



Let ABC be a triangle, and CE a line drawn from the vertex to the middle of the baſe AB: then will twice the ſum of the ſquares of CE, EA be equal to the ſum of the ſquares of AC, CB.

For on AB, produced if neceſſary, let fall the perpendicular CD (I. 12.)

Then, becauſe AEC is an obtuſe angle, the ſquare of AC is equal to the ſquares of AE, EC together with twice the rectangle of AE, ED (II. 17.)

And, becauſe BEC is an acute angle, the ſquare of CB together with twice the rectangle of BE, ED is equal to the ſquares of BE, EC (II. 18.)

And ſince AE is equal to EB (*by Conſt.*), the ſquare of BC together with twice the rectangle of AE, ED is equal to the ſquares of AE, EC.

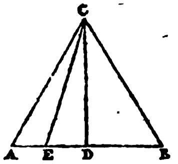
But if equals be added to equals, the wholes will be equal; whence the ſquares of AC, CB, together with twice the rectangle of AE, ED, are equal to twice the ſquares of AE, EC, together with twice the rectangle of AE, ED.

And, if twice the rectangle of AE, EC, which is common, be taken away, the ſum of the ſquares of AC, CB will be equal to twice the ſum of the ſquares of AE, EC.

Q.E.D.

PROP. XX. THEOREM.

In an iſoſceles triangle, the ſquare of a line drawn from the vertex to any point in the baſe, together with the rectangle of the ſegments of the baſe, is equal to the ſquare of one of the equal ſides of the triangle.



Let ABC be an iſoſceles triangle, and CE a line drawn from the vertex to any point in the baſe AB; then will the ſquare of CE, together with the rectangle of AE, EB be equal to the ſquare of AC or CB.

For biſect the baſe AB in D (I. 10.), and join the points C, D.

Then, ſince AC is equal to CB, AD to DB, and CD is common to each of the triangles ACD, BCD, the angle CDA will be equal to the angle CDB (I. 7.); and conſequently CD will be perpendicular to AB (*Def.* 8, 9.)

And, becauſe ACE is a triangle, and CD is the perpendicular, the difference of the ſquares of AC, CE is equal to the difference of the ſquares of AD, DE (II. 16.)

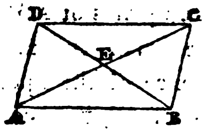
But, ſince BE is the ſum of AD and DE, and AE is their difference, the difference of the ſquares of AD, DE is equal to the rectangle of AE, EB; conſequently, the difference of the ſquares of AC, CE is alſo equal to the rectangle AE, EB.

And if, to each of theſe equals, there be added the ſquare of CE, the ſquare of AC will be equal to the ſquare of CE together with the rectangle of AE, EB.

Q. E. D.

PROP. XXI. THEOREM.

The diagonals of any parallelogram biſect each other, and the ſum of their ſquares is equal to the ſum of the ſquares of the four ſides of the parallelogram.



Let ABCD be a parallelogram, whoſe diagonals AC, BD interſect each other in E; then will AE be equal to EC, and BE to ED; and the ſum of the ſquares of AC, BD will be equal to the ſum of the ſquares of AB, BC, CD and DA.

For ſince AB, DC are parallel, and AC, BD interſect them, the angle DCE will be equal to the angle EAB (I. 24.), and the angle CDE to the angle EBA (I. 24.)

The angle DEC is likewiſe equal to the angle AEB (I. 15.), and the ſide DC to the ſide AB (I. 30.); conſequently DE is alſo equal to EB, and CE to EA (I. 21.)

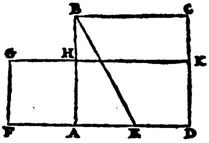
Again, ſince DB is biſected in E, the ſum of the ſquares of DC, CB will be equal to twice the ſum of the ſquares of DE, EC (II. 19.)

And, becauſe DC is equal to AB, and CB to DA (I. 30.) the ſum of the ſquares of AB, CB, DC and DA are equal to four times the ſum of the ſquares of DE, EC.

But four times the ſquare of DE is equal to the ſquare of BD (II. 11. *Cor.*), and four times the ſquare of EC is equal to the ſquare of AC; whence the ſum of the ſquares of AC, BD are equal to the ſum of the ſquares of AB, BC, CD and DA. Q.E.D.

PROP. XXII. PROBLEM.

To divide a given right line into two parts, ſo that the rectangle contained by the whole line and one of the parts, ſhall be equal to the ſquare of the other part.



Let AB be the given right line; it is required to divide it into two parts, ſo that the rectangle of the whole line and one of the parts ſhall be equal to the ſquare of the other part.

Upon AB deſcribe the ſquare AC (II. 1.); and biſect the ſide of it AD in E (I. 10.)

Join the points B, E; and, in EA produced, take EF equal to EB (I. 3.); and upon AF deſcribe the ſquare FH (II. 1.)

Then will AB be divided in H ſo, that the rectangle of AB, BH, will be equal to the ſquare of AH.

For, ſince DF is equal to the ſum of EB and ED, or its equal EA, and AF is equal to their difference, the rectangle of DF, FA is equal to the difference of the ſquares of EB, EA (II. 13.)

But the rectangle of DF, FA is equal to DG, becauſe FA is equal to FG (II. *Def.* 2.); and the difference of the ſquares of EB, EA is equal to the ſquare of AB (II. 14. *Cor.*); whence DG is equal to AC.

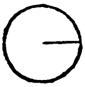
And, if from each of theſe equals, the part DH, which is common to both, be taken away, the remainder AG will be equal to the remainder HC.

But HC is the rectangle of AB, BH; for AB is equal to BC; and AG is the ſquare of AH; therefore the right line AB is divided in H ſo, that the rectangle of AB, BH is equal to the ſquare of AH, which was to be done.

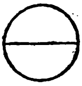
BOOK III.

DEFINITIONS.

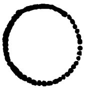
1. A radius of a circle, is a right line drawn from the centre to the circumference.



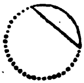
2. A diameter of a circle, is a right line drawn through the centre and terminated both ways by the circumference.



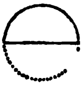
3. An arc of a circle, is any part of its periphery, or circumference.



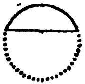
4. The chord, or ſubtenſe, of an arc, is a right line joining the two extremities of that arc.



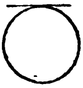
5. A ſemicircle, is a figure contained under any diameter and the part of the circumference cut off by that diameter.



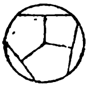
6. A ſegment of a circle, is a figure contained under any arc and the chord of that arc.



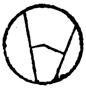
7. A tangent to a circle, is a right line which paſſes through a point in the circumference without cutting it.



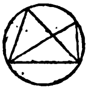
8. Right lines, or chords, are ſaid to be equally diſtant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.



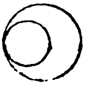
9. And the right line on which the greater perpendicular falls, is ſaid to be farther from the centre.



10. An angle in a ſegment, is that which is contained by two right lines, drawn from any point in the arc of the ſegment, to the two extremities of the chord of that arc.



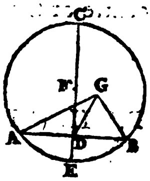
11. One circle is ſaid to touch another, when it paſſes through a point in its circumference without cutting it.



PROP.

PROP. I. PROBLEM.

To find the centre of a given circle.



Let ABC be the given circle; it is required to find its centre.

Draw any chord AB, and biſect it in D (I. 10.); and through the point D draw CE at right angles to AB (I. 11.), and biſect it in F: then will the point F be the centre of the circle.

For if it be not, ſome other point muſt be the centre, either in the line EC, or out of it.

But it cannot be any other point in the line EC, for if it were, two lines drawn from the centre of the circle to its circumference would be unequal, which is abſurd.

Neither can it be any point out of that line; for if it can, let G be that point; and join GA, GD and GB.

Then, becauſe GA is equal to GB (I. *Def.* 13.), AD to DB (*by Conſt.*), and GD common to each of the triangles AGD, BGD, the angle ADG will be equal to the angle BDG (I. 7.)

But when one line falls upon another, and makes the adjacent angles equal, thoſe angles are, each of them, right angles (I. *Def.* 8 *and* 9.)

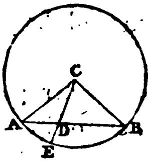
The angle ADG, therefore, is equal to the angle ADC (I. 8.), the whole to the part, which is abſurd; conſequently no point but F can be the centre of the circle.

Q.E.D.

COROLL. If any chord of a circle be biſected, a right line drawn through that point, perpendicular to the chord, will paſs through the centre of the circle.

PROP. II. THEOREM.

If any two points be taken in the circumference of a circle, the chord, or right line which joins them, will fall wholly within the circle.



Let ABE be a circle, and A, B any two points in the circumference; then will the right line AB, which joins thoſe points, fall wholly within the circle.

For find C, the centre of the circle ABE (III. 1.), and join C, A, C, B; and through any point D, in AB, draw the right line CE, cutting the circumference in E.

Then, becauſe CA is equal to CB (I. *Def.* 13.), the angle CAB will be equal to the angle CBA (I. 5.)

And, ſince the outward angle CDB of the triangle ACD, is greater than the inward oppoſite angle CAB (I. 16.), it will alſo be greater than the angle CBA.

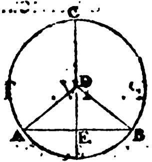
But the greater ſide of every triangle is oppoſite to the greater angle (I. 17.); whence CB, or its equal CE, will be greater than CD.

The point D, therefore, falls within the circle; and the ſame may be ſhewn of any other point in AB; conſequently the whole line AB muſt fall within the circle.

Q.E.D.

PROP. III. THEOREM.

If a right line, which paſſes through the centre of a circle, biſect a chord, it will be perpendicular to it; and if it be perpendicular to the chord, it will biſect it.



Let ABC be a circle, and CE a right line which paſſes through the centre D, and biſects the chord AB in E; then will CE be perpendicular to AB.

For join the points AD, DB:

Then, becauſe AD is equal to DB (II. *Def.* 13.), AE to EB (*by Hyp.*), and ED common to each of the triangles ADE, BDE, the angle DEA will be equal to the angle DEB (I. 7.)

But one line is ſaid to be perpendicular to another, when it makes the angles on both ſides of it equal to each other (I. *Def.* 8.); conſequently CE is perpendicular to the chord AB.

Again, let the right line DE be drawn from the centre D, perpendicular to the chord AB; then will AB be biſected in the point E.

For join the points AD, DB, as before:

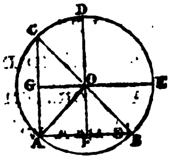
Then, ſince the angle DAB is equal to the angle DBA (I. 5.), and the angle AED to the angle DEB, (being each of them right angles) the angle ADE will alſo be equal to the angle EDB (I. 28. *Cor.* 1.)

And, becauſe the triangles DEA, DEB are mutually equiangular, and have the ſide DE common, the ſide AE will alſo be equal to the ſide EB (I. 21.); whence AB is biſected in the point E, as was to be ſhewn.

COROLL. If a right line be drawn from the vertex of an iſoſceles triangle, to the middle of the baſe, it will be perpendicular to it; and if it be perpendicular to the baſe, it will biſect both it and the vertical angle.

PROP. IV. THEOREM.

If more than two equal right lines can be drawn from any point in a circle to the circumference, that point will be the centre.



Let ABDC be a circle, and O point within it; then if any three right lines OA, OB, OC, drawn from the point O to the circumference, be equal to each other, that point will be the centre.

For draw the lines AB, AC, and biſect them in the points F, G (I. 10.); and through the centre O, draw FD, GE, cutting the circumference in D and E.

Then, ſince AF is equal to FB (*by Conſt.*), AO to OB (*by Hyp.*), and OF common to each of the triangles AOF, BOF, the angle AFO will be equal to the angle BFO (I. 7.)

And becauſe the right line OF falls upon the right line AB, and makes the adjacent angles equal to each other, DF will be perpendicular to AB (I. *Def.* 8.)

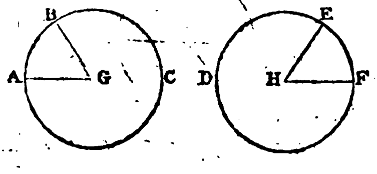
But when a right line biſects any chord at right angles, it paſſes through the centre of the circle (III. 1. *Cor.*); whence the centre muſt be ſomewhere in the line FD.

And, in the ſame manner, it may be ſhewn, that the centre muſt be ſomewhere in the line GE.

But the lines FD, GE have no other point but O which is common to them both; therefore O is the centre of the circle ABD, as was to be ſhewn.

PROP. V. THEOREM.

Circles of equal radii are equal to each other; and if the circles are equal, the radii will be equal.



Let ABC, DEF be two circles, of which the radii GA, GB are equal to the radii HF, HE; then will the circle ABC be equal to the circle DEF.

For conceive the circle DEF to be applied to the circle ABC, ſo that the centre H may coincide with the centre G.

Then, ſince the radii HF, HE are equal to the radii GA, GB (*by Hyp.*), the points F, E will fall in the circumference of the circle ABC (I. *Def.* 13.); and the ſame may be ſhewn of any other point D.

But ſince any number of points, taken in the circumference of the circle DEF, fall in the circumference of the circle ABC, the two circumferences muſt coincide, and conſequently the circles are equal to each other.

Again, let the circle ABC be equal to the circle DEF; then will the radii GA, GB be equal to the radii HF, HE.

For if they be not equal, they muſt be either greater or leſs: let them be greater; and apply the circles to each other as before.

Then, ſince the radii GA, GB are greater than the radii HF, HE, the points F, E will fall within the circle ABC; and the ſame may be ſhewn of any other point D.

But, ſince any number of points, taken in the circumference of the circle DEF, fall within the circle ABC, the whole circle DEF muſt, alſo, fall within the circle ABC.

The circle DEF is, therefore, leſs than the circle ABC, and equal to it at the ſame time (*by Hyp.*), which is abſurd: whence the radii GA, GB are not greater than the radii HF, HE.

And in the ſame manner it may be ſhewn that they cannot be leſs; conſequently they are equal to each other.

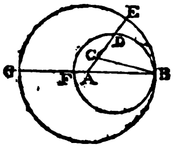
Q.E.D.

COROLL. Equal circles, or ſuch as have equal radii, or diameters, have equal circumferences.

PROP.

PROP. VI. THEOREM.

If two circles touch each other internally, the centres of the circles and the point of contact will be all in the ſame right line.



Let the two circles BEG, BDF touch each other internally at the point B; then will the centres of thoſe circles and the point B be in the ſame right line.

For let A be the centre of the circle BEG, and draw the diameter GB.

And if the centre of the circle BDF be not in GB, let, if poſſible, ſome point C, out of that line, be the centre; and join A, C, C, B; and produce AC to cut the circles in D and E.

Then, ſince ACB is a triangle, the ſides AC, CB, taken together, are greater than the ſide AB (I. 18.), or its equal AE.

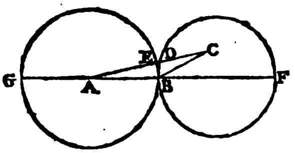
And if, from theſe equals, the part AC, which is common, be taken away, the remainder CB will be greater than the remainder CE.

But, ſince C is the centre of the circle BDF (*by Hyp.*), CB is equal to CD (I. *Def.* 13.); whence CD will alſo be greater than CE, which is impoſſible.

The point C, therefore, cannot be the centre of the circle BDF; and the ſame may be ſhewn of any other point out of the line AB. Q. E. D.

PROP. VII. THEOREM.

If two circles touch each other externally, the centres of the circles and the point of contact will be all in the ſame right line.



Let the two circles BEG, BDF touch each other externally at the point B; then will the centres of thoſe circles and the point B, be in the ſame right line.

For, let A be the centre of the circle BEG, and draw the diameter GB, which produce till it cuts the circle BDF in F.

And, if the centre of the circle BDF be not in the line AF, let, if poſſible, ſome point C, out of that line, be the centre; and join C, A, C, B.

Then, ſince A is the centre of the circle BEG, AE is equal to AB (I. *Def.* 13.)

And becauſe C is the centre of the circle BDF (*by Hyp.*) CD is equal to CB (I. *Def.* 13.)

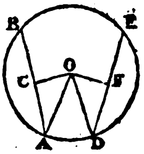
But AB, BC, together, are greater than AC (I. 18.); therefore AE, CD, together, are alſo greater than AC; which is abſurd.

The point C, therefore, cannot be the centre of the circle BDF; and the ſame may be ſhewn of any other point out of the line AF. Q. E. D.

PROP.

PROP. VIII. THEOREM.

Any two chords in a circle, which are equally diſtant from the centre, are equal to each other; and if they be equal to each other, they will be equally diſtant from the centre.



Let ABED be a circle, whoſe centre is O; then will any two chords AB, DE, which are equally diſtant from O, be equal to each other.

For join the points AO, OD, and let fall the perpendiculars OC, OF (I. 12.)

Then, ſince a right line, drawn from the centre of a circle, at right angles to any chord, biſects it (III. 3.), AC will be equal to CB, and DF to FE.

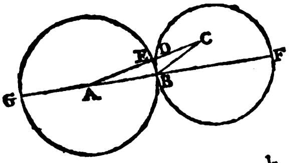
And, becauſe the angles ACO, DFO are right angles, the ſquares of AC, CO will be equal to the ſquare of AO (II. 14.), and the ſquares of DF, FO to the ſquare of OD.

But the ſquare of AO is equal to the ſquare of OD (II. 2.); conſequently the ſquares of AC, CO will be equal to the ſquares of DF, FO.

And ſince OC is equal to OF (III. *Def.* 8.), the ſquare of OC will be equal to the ſquare of OF (II. 2.); whence

PROP. VII. THEOREM.

If two circles touch each other externally, the centres of the circles and the point of contact will be all in the ſame right line.



Let the two circles BEG, BDF touch each other externally at the point B; then will the centres of thoſe circl and the point B, be in the ſame right line.

For, let A be the centre of the circle BEG, and dr the diameter GB, which produce till it cuts the cir BDF in F.

And, if the centre of the circle BDF be not in the AF, let, if poſſible, ſome point C, out of that line, b centre; and join C, A, C, B.

Then, ſince A is the centre of the circle BEG, equal to AB (I. *Def.* 13.)

And becauſe C is the centre of the circle BDF (b CD is equal to CB (I. *Def.* 13.)

But AB, BC, together, are greater than AC (I therefore AE, CD, together, are alſo greater th which is abſurd.

The point C, therefore, cannot be the cent circle BDF; and the ſame may be ſhewn of point out of the line AF.

the remaining ſquare of AC will alſo be equal to the remaining ſquare of DF; or AC equal to DF (II. 3.), and AB to DE (I. *Ax.* 6.)

Again, let the chord AB be equal to the chord DE; then will OC, OF, or their diſtances from the centre, be equal to each other.

For the ſquares of AC, CO are equal to the ſquare of OA (II. 14.), and the ſquares of DF, FO to the ſquare of OD.

But the ſquare of OA is equal to the ſquare of OD (II. 2.); therefore the ſquares of AC, CO are equal to the ſquares of DF, FO.

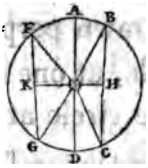
And ſince AC is the half of AB (III. 3.), and DF is the half of DE (III. 3.), the ſquare of AC is equal to the ſquare of DF (II. 2.)

The remaining ſquare of CO is, therefore, equal to the remaining ſquare of FO; and conſequently CO is equal to FO (II. 3.), as was to be ſhewn.

COROLL. If two right angled triangles, having equal hypotenuſes, have two other ſides alſo equal, the remaining ſides will likewiſe be equal, and the triangles will be equal in all reſpects.

PROP. IX. THEOREM.

A diameter is the greateſt right line that can be drawn in a circle, and, of the reſt, that which is nearer the centre is greater than that which is more remote.



Let ABCD be a circle, of which the diameter is AD, and the centre O; then if BC be nearer the centre than FG, AD will be greater than BC, and BC than FG.

For draw OH, OK perpendicular to BC, FG (I. 12.), and join OB, OC, OG and OF.

Then, becauſe OA is equal to OB (I. *Def.* 13.), and OD to OC, AD is equal to OB and OC taken together.

But OB, OC, taken together, are greater than BC (I. 18.); therefore AD is alſo greater than BC.

Again, the ſquares of OH, HB are equal to the ſquare of OB (II. 14.), and the ſquares of OK, KF to the ſquare of OF.

But the ſquare of OB is equal to the ſquare of OF (II. 2.); whence the ſquares of OH, HB are equal to the ſquares of OK, KF.

And ſince FG is farther from the centre than BC (*by Hyp.*), OK will be greater than OH (III. *Def.* 9), and the ſquare of OK than the ſquare of OH (II. 4.)

The

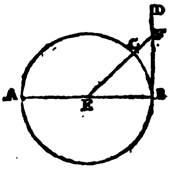
The remaining ſquare of HB, therefore, is greater than the remaining ſquare of KF, and HB greater than KF (II. 4.)

But BC is the double of BH, and FG is the double of FK (III. 3.); conſequently BC is alſo greater than FG.

Q. E. D.

PROP. X. THEOREM.

A right line drawn perpendicular to the diameter of a circle, at one of its extremities, is a tangent to the circle at that point.



Let ABC be a circle whoſe centre is E, and diameter AB; then if DB be drawn perpendicular to AB, it will be a tangent to the circle at the point B.

For in BD take any point F, and draw EF, cutting the circumference of the circle in C.

Then, ſince the angle EBD is a right angle (*by Hyp.*), the angles BEF, EFB, will be each of them leſs than a right angle (I. 28.)

And, becauſe the greater ſide of every triangle is oppoſite to the greater angle (I. 17.), the ſide EF is greater than the ſide EB, or its equal EC.

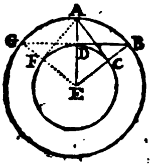
But ſince EF is greater than EC, the point F will fall without the circle ABC; and the ſame may be ſhewn of any other point in BD, except B.

The line BD, therefore, cannot cut the circle, but muſt fall wholly without it, and be a tangent to it at the point B, as was to be ſhewn.

SCHOLIUM. A right line cannot touch a circle in more than one point, for if it met it in two points it would fall wholly within the circle (III. 2.)

PROP. XI. THEOREM.

From a given point to draw a tangent to a given circle.



Let A be the given point, and FDC the given circle; it is required from the point A to draw a tangent to the circle FDC.

Find E, the centre of the circle FDC (III. 1.), and join EA; and from the point E, at the diſtance EA, deſcribe the circle GAB.

Through the point D, draw DB at right angles to EA (I. 11.), and join EB, AC; and AC will be the tangent required.

For, ſince E is the centre of the circles FDC, GAB, EA is equal to EB, and ED to EC.

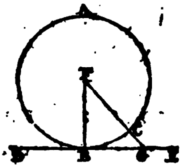
And, becauſe the two ſides EA, EC, of the triangle EAC, are equal to the two ſides EB, ED, of the triangle EBD, and the angle E is common, the angle ECA will alſo be equal to the angle EDB (I. 4.)

But the angle EDB being a right angle, the angle ECA is alſo a right angle; therefore ſince AC is perpendicular to the diameter EC, it will touch the circle FDC, and be a tangent to it at the point C (III. 10.)

Q. E. I.

PROP. XII. THEOREM.

If a right line be a tangent to a circle, and another right line be drawn from the centre to the point of contact, it will be perpendicular to the tangent.



Let the right line DE be a tangent to the circle ABC at the point B, and, from the centre F, draw the right line FB; then will FB be perpendicular to DE.

For if it be not, let, if poſſible, ſome other right line FG be perpendicular to DE.

Then, becauſe the angle FGB is a right angle (*by Hyp.*), the angle FBG will be leſs than a right angle (I. 28.)

And, ſince the greater ſide of every triangle is oppoſite to the greater angle (I. 17.), the ſide FB will be greater than the ſide FG.

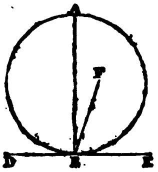
But FB is equal to FC; therefore FC will alſo be greater than FG, a part greater than the whole, which is impoſſible.

The line FG, therefore, cannot be perpendicular to DE; and the ſame may be demonſtrated of any other line but FB; conſequently FB is perpendicular to DE.

Q. E. D.

PROP. XIII. THEOREM.

If a right line be a tangent to a circle, and another right line be drawn at right angles to it, from the point of contact, it will paſs through the centre of the circle.



Let the right line DE be a tangent to the circle ACB at the point B; then if AB be drawn at right angles to DE, from the point of contact B, it will paſs through the centre of the circle.

For if it does not, let F, if poſſible, be the centre of the circle; and join FB.

Then, ſince DE is a tangent to the circle, and FB is a right line drawn from the centre to the point of contact, the angle FBE is a right angle (III. 12.)

But the angle ABE is alſo a right angle, by conſtruction; whence the angle FBE is equal to the angle ABE; the leſs to the greater, which is impoſſible.

The point F, therefore, is not the centre; and the ſame may be ſhewn of any other point which is out of the line AB; conſequently AB muſt paſs through the centre of the circle, as was to be ſhewn.

PROP.

Then, ſince an angle at the centre of a circle is double to that at the circumference (III. 14.), the angle BAC will be half the angle BEC.

And, for the ſame reaſon, the angle BDC will, alſo, be half the angle BEC.

But things which are halves of the ſame thing are equal to each other; conſequently the angle BAC is equal to the angle BDC.

Again, let the ſegment BADC be not greater than a ſemicircle;

Then, ſince the right lines BD, AC interſect each other in F, the angle BFA will be equal to the oppoſite angle DFC (I. 15.)

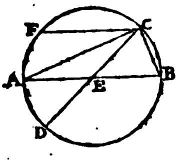
And becauſe the ſegment ABCD is greater than a ſemicircle, the angles ABD, ACD, which ſtand in that ſegment, are equal to each other (III. 15.)

But ſince the two angles BFA, ABF of the triangle FBA, are equal to the two angles DFC, FCD of the triangle DCF, the remaining angle BAF, or BAC, will alſo be equal to the remaining angle FDC, or BDC.

Q. E. D.

PROP. XVI. THEOREM.

An angle in a ſemicircle is a right angle.



Let ABC be a ſemicircle; then will any angle ACB, in that ſemicircle, be a right angle.

For, find the centre of the circle E (III. 1.) and draw the diameter CED.

Then, becauſe an angle at the centre of a circle is double to that at the circumference (III. 14.) the angle AED will be double the angle ACD.

And, for the ſame reaſon, the angle BED will be double the angle BCD.

The angles AED, BED, therefore, taken together, are double the whole angle ACB.

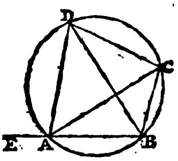
But the angles AED, BED, are, together, equal to two right angles (I. 13.); conſequently the angle ACB will be equal to one right angle. Q. E. D.

COROLL. The angle BAC, which ſtands in a ſegment greater than a ſemi-circle, is leſs than a right angle (I. 28.):

And the angle BCF, which ſtands in a ſegment leſs than a ſemi-circle, is greater than a right angle.

PROP. XVII. THEOREM.

The oppoſite angles of any quadrilateral figure, inſcribed in a circle, are equal to two right angles.



Let ABCD be a quadrilateral, inſcribed in the circle ADCB; then will the oppoſite angles BAD, BCD, taken together, be equal to two right angles.

For, draw the diagonals AC, BD, and produce the ſide BA to E.

Then, becauſe the outward angle of any triangle, is equal to the two inward oppoſite angles taken together (I. 28.), the angle EAD will be equal to the angles ABD, ADB.

And, becauſe all angles in the ſame ſegment of a circle are equal to each other (III. 15.), the angle ABD will be equal to the angle ACD, and the angle ADB to the angle ACB.

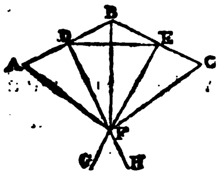
The angle EAD, therefore, which is equal to the angles ABD, ADB, taken together, will alſo be equal to the angles ACD, ACB, taken together, or to the whole angle BCD.

But the angles EAD, BAD, taken together, are equal to two right angles (I. 13.); conſequently the angles BCD, BAD, taken together, will alſo be equal to two right angles. Q. E. D.

COROLL. If any ſide AB, of the quadrilateral ABCD, be produced, the outward angle EAD will be equal to the inward oppoſite angle BCD.

PROP. XVIII. THEOREM.

Through any three points, not ſituate in the ſame right line, to deſcribe the circumference of a circle.



Let A, B, C, be any three points, not ſituate in the ſame right line; it is required to deſcribe the circumference of a circle through thoſe points.

Draw the right lines AB, BC, and biſect them with the perpendiculars DH, EG (I. 10 *and* 11.); and join DE.

Then, becauſe the angles FED, FDE are leſs than two right angles, the lines DH, EG will meet each other, in ſome point F (*Cor.* I. 25.); and that point will be the centre of the circle required.

For, draw the lines FA, FB and FC,

Then, ſince AD is equal to DB, DF common, and the angle ADF equal to the angle FDB (I. 8.), the ſide FA will alſo be equal to the ſide FB (I. 4.)

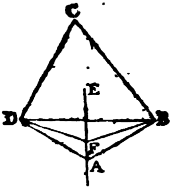
And, in the ſame manner, it may be ſhewn, that the ſide FC is alſo equal to the ſide FB.

The lines FA, FB and FC, are, therefore, all equal to each other; and conſequently F is the centre of a circle which will paſs through the points A, B and C, as was to be ſhewn.

SCHO. If the ſegment of a circle be given, and any three points be taken in the circumference, the centre of the circle may be found, as above.

PROP. XIX. THEOREM.

If the oppoſite angles of a quadrilateral, taken together, be equal to two right angles, a circle may be deſcribed about that quadrilateral.



Let ABCD be a quadrilateral, whoſe oppoſite angles DCB, DAB are, together, equal to two right angles: then may a circle be deſcribed about that quadrilateral.

For ſince the circumference of a circle may be deſcribed through any three points (III. 18.), let E be the centre of a circle which paſſes through the points D, C, B; and draw the indefinite right line EFA.

And if the circle does not paſs through the fourth point A, let it paſs, if poſſible, through ſome other point F, in the line EA, and draw the lines DF, FB, and BD.

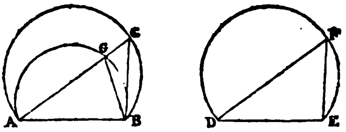
Then, ſince the oppoſite angles BFD, DCB are, together, equal to two right angles (III. 17.), and the angles BAD, DCB are alſo equal to two right angles (*by Hyp.*), the angles BFD, DCB will be equal to the angles BAD, DCB.

And if, from each of theſe equals, there be taken the angle DCB, which is common to both, the remaining angle BAD will be equal to the remaining angle BFD; or, which is the ſame thing, the two angles DFE, EFB will be equal to the two angles DAE, EAB, which is impoſſible (I. 16.)

The circumference of the circle, therefore, cannot paſs through the point F; and the ſame may be demonſtrated of any other point in the line EA, except the point A; whence a circle may be deſcribed about the quadrilateral ABCD, as was to be ſhewn.

PROP. XX. THEOREM.

Segments of circles, which ſtand upon equal chords, and contain equal angles, are equal to each other.



Let ACB, DFE be two ſegments of circles, which ſtand upon the equal chords AB, DE, and contain equal angles; then will thoſe ſegments be equal to each other.

For let the ſegment DFE be applied to the ſegment ACB, ſo that the point D may fall upon the point A, and the line DE upon the line AB.

Then, ſince DE is equal to AB (*by Hyp.*), the point E will fall upon the point B, and the two ſegments will coincide with each other.

For if they do not, there muſt be ſome point, in the circumference of one of them, which will fall either within or without the other.

Let the point F, in the circumference of the circle DFE, be that point; which ſuppoſe to fall at G within the circle ACB; and draw the lines AGC, BC and BG.

Then, ſince the outward angle AGB, of the triangle BCG, is greater than the inward oppoſite angle GCB, it will alſo be greater than the angle DFE, which is equal to GCB, or ACB (*by Hyp.*)

But the angle AGB is alſo equal to the angle DFE, becauſe the ſegments in which they ſtand are identical; whence they are equal and unequal at the ſame time, which is abſurd.

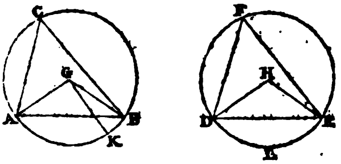
The point F, therefore, cannot fall within the circle ACB; and in the ſame manner it may be ſhewn that it cannot fall without it; conſequently the ſegments muſt coincide, and be equal to each other. Q. E. D.

COROLL. Segments of circles, which ſtand upon equal chords, and contain equal angles, have equal circumferences.

PROP.

PROP. XXI. THEOREM.

In equal circles, equal angles ſtand upon equal arcs, whether they be at the centres or circumferences; and if the arcs be equal, the angles will be equal.



Let ABC, DEF be two equal circles, having the angles AGB, DHE, at their centres, equal to each other, as alſo the angles ACB, DFE, at their circumferences; then will the arc AKB be equal to the arc DLE.

For, join the points AB, DE: then, ſince the circles are equal to each other (*by Hyp.*), their radii and circumferences will alſo be equal (III. 5.)

And, ſince the two ſides AG, GB of the triangle ABG, are equal to the two ſides DH, HE of the triangle DEH, and the angle AGB to the angle DHE (*by Hyp.*), their baſes AB, DE will likewiſe be equal (I. 4.)

The chord AB, therefore, being equal to the chord DE, and the angle ACB to the angle DFE (*by Hyp.*), the arc BCA will alſo be equal to the arc EFD (*Cor.* III. 20.)

But ſince the whole circumference of the circle ABC, is equal to the whole circumference of the circle DEF, and the arc BCA to the arc EFD, the arc AKB will alſo be equal to the arc DLE.

Again, let the arc AKB be equal to the arc DLE; then will the angle AGB be equal to the angle DHE, and the angle ACB to the angle DFE.

For, if AGB be not equal to DHE, one of them muſt be greater than the other; let AGB be the greater; and make the angle AGK equal to DHE (I. 20.)

Then, ſince equal angles ſtand upon equal arcs (III. 21.), the arc AK will be equal to the arc DLE.

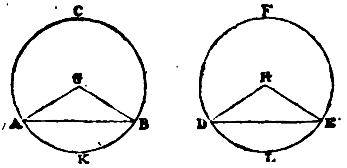
But the arc DLE is equal to the arc AKB (*by Hyp.*); whence the arc AK is alſo equal to the arc AKB; the leſs to the greater, which is impoſſible.

The angle AGB, therefore, is not greater than the angle DHE; and in the ſame manner it may be proved that it cannot be leſs; conſequently they are equal to each other.

And ſince angles at the centre are double to thoſe at the circumference, the angle ACB will alſo be equal to the angle DFE. Q.E.D.

PROP. XXII. THEOREM.

In equal circles, equal chords ſubtend equal arcs; the greater equal to the greater, and the leſs to the leſs; and if the chords be equal the arcs will be equal.



Let ABC, DEF be two equal circles, in which the chord AB is equal to the chord DE; then will the arc

ACB be equal to the arc DFE, and the arc AKB to the arc DLE.

For, find G, H, the centres of the circles (III. 1.), and join GA, GB, HD and HE.

Then, ſince the circles are equal to each other (*by Hyp.*) their radii and circumferences will alſo be equal (III. 5.)

And, ſince the ſides AG, GB are equal to the ſides DH, HE, and the baſe AB to the baſe DE (*by Hyp.*), the angle AGB will alſo be equal to the angle DHE (I. 22.)

But equal angles, at the centres of equal circles, ſtand upon equal arcs (III. 21.); therefore the arc AKB is equal to the arc DLE.

And ſince the whole circumference of the circle ABC is equal to the whole circumference of the circle DEF, and the arc AKB to the arc DLE, the arc ACB will alſo be equal to the arc DFE.

Again, let ABC, DEF be two equal circles, of which the arc AKB is equal to the arc DLE; then will the chord AB be equal to the chord DE,

For let G, H be the centres of the circles, found as before; and join AG, GB, DH and HE.

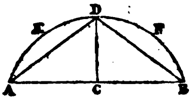
Then, ſince the circles are equal to each other (*by Hyp.*), the radii AG, GB will be equal to the radii DH, HE (III. 5.)

And becauſe the arc AKB is equal to the arc DLE (*by Hyp.*), the angles AGB, DHE, at the centres, will be equal (III. 21.)

But, ſince the two ſides AG, GB are equal to the two ſides DH, HE, and the angle AGB to the angle DHE, the baſe AB will alſo be equal to the baſe DE (I. 4.) Q.E.D.

PROP. XXIII. PROBLEM.

To biſect a given arc, that is, to divide it into two equal parts.



Let ADB be the given arc; it is required to divide it into two equal parts.

Draw the right line AB, which biſect in C (I. 10.); and, from the point C, erect the perpendicular CD (I. 11.); then will the arc ADB be biſected in the point D, as was required.

For, join the points AD, DB: then, ſince the two ſides AC, CD, of the triangle ADC, are equal to the two ſides BC, CD of the triangle BDC, and the angle ACD to the angle BCD (I. 8.), the baſe AD will be equal to the baſe DB (I. 4.)

And, becauſe DC, or DC produced, paſſes through the centre of the circle (III. 1 *Cor.*), the ſegments ADE, DBF will be each of them leſs than a ſemicircle.

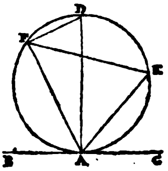
But equal chords are ſubtended by equal arcs, the greater equal to the greater, and the leſs to the leſs (III, 22.); whence the chord AD being equal to the chord DB, the arc AED will be equal to the arc DFB. Q. E. I.

SCHOLIUM. An arc of a circle cannot, in general, be triſected, or divided into three equal parts, by any known method, which is purely geometrical.

PROP.

PROP. XXIV. THEOREM.

The angle formed by a tangent to a circle and a chord drawn from the point of contact, is equal to the angle in the alternate ſegment.



Let BC be a tangent to the circle AFDE, and AF a chord drawn from the point of contact; then will the angle CAE be equal to the angle AFE in the alternate ſegment.

For draw the diameter AD (III. 1.) and join the points F, D:

Then, becauſe BC is a tangent to the circle, and AD is a line drawn through the centre, from the point of contact, the angle DAC will be a right angle (III. 12.)

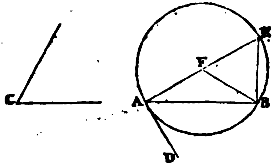
And, becauſe AFD is a ſemi-circle, the angle DFA will alſo be a right angle (III. 16.), and equal to the angle DAC.

But ſince all angles in the ſame ſegment of a circle are equal to each other (III. 15.), the angle DFE will be equal to the angle DAE.

If, therefore, from the equal angles DAC, DFA, there be taken the equal angles DAE, DFE, the remaining angle CAE will be equal to the remaining angle AFE. Q.E.D.

PROP. XXV. PROBLEM.

Upon a given right line to deſcribe a ſegment of a circle, that ſhall contain an angle equal to a given rectilineal angle.



Let AB be the given right line, and C the given rectilineal angle; it is required to deſcribe a ſegment of a circle upon the line AB that ſhall contain an angle equal to C.

Make the angle BAD equal to C (I. 20.), and, from the point A, draw AE at right angles to AD (I. 11.) and make the angle ABF equal to the angle FAB (I. 20,)

Then, ſince the angles ABF, FAB are equal to each other, and leſs than two right angles, the ſides AF, FB will meet, and be equal to each other (I. 25 *Cor. and* I. 6.)

From the point F, therefore, at the diſtance FA, or FB, deſcribe the circle AEB, and ABEA will be the ſegment required.

For let AF be produced to cut the circle in E; and join the points E, B.

Then, becauſe AD is perpendicular to the diameter AE, it will be a tangent to the circle at the point A (III. 10.)

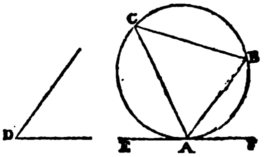
And, becauſe AD touches the circle, and AB is a chord drawn from the point of contact, the angle BAD will be equal to the angle AEB in the alternate ſegment (III. 24.)

But the angle BAD is equal to the angle C, by conſtruction; conſequently the angle AEB is alſo equal to the angle C. Q.E.I.

SCHOLIUM. When the given angle is a right angle, a ſemi-circle deſcribed upon the given line will be the ſegment required (III. 16.)

PROP. XXVI. PROBLEM.

To cut off a ſegment from a given circle, that ſhall contain an angle equal to a given rectilineal angle.



Let ABC be a given circle, and D a given rectilineal angle; it is required to cut off a ſegment from the circle ABC, that ſhall contain an angle equal to D.

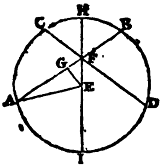
Draw the right line EF, to touch the circle ABC in the point A (III. 10.), and make the angle FAB equal to the angle D (I. 20.); then will ABCA be the ſegment required.

For, ſince EF is a tangent to the circle, and AB is a chord drawn from the point of contact, the angle FAB will be equal to the angle ACB in the alternate ſegment (III. 24.)

But the angle FAB is equal to the angle D, by conſtruction; conſequently the angle ACB, in the ſegment ABCA, is alſo equal to the angle D. Q.E.I.

PROP. XXVII. THEOREM.

If two right lines in a circle interſect each other, the rectangle contained under the ſegments of the one, will be equal to the rectangle contained under the ſegments of the other.



Let AB, CD be any two right lines, in the circle ACBD, interſecting each other in the point F; then will the rectangle contained under the parts AF, FB of the one, be equal to the rectangle contained under the parts CF, FD of the other.

For, through the point F, draw the diameter HI (III. 1.); and, from the centre E, draw EG at right angles to AB (I. 12.), and join AE:

Then, ſince AEF is a triangle, and the perpendicular EG divides the chord AB into two equal parts (III. 3.), the line FB will be equal to the difference of the ſegments AG, GF.

And, becauſe E is the centre of the circle, and AE is equal to EI or EH, the line FI will be equal to the ſum of the ſides AE and EF; and FH will be equal to their difference.

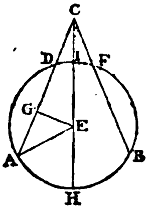
But the rectangle under the ſum and difference of the two ſides of any triangle, is equal to the rectangle under the baſe and the difference of the ſegments of the baſe (*Cor.* II. 16.); whence the rectangle of IF, FH is equal to the rectangle of AF, FB.

And, in the ſame manner, it may be proved, that the rectangle of IF, FH, is equal to the rectangle of DF, FC: conſequently the rectangle of AF, FB is alſo equal to the rectangle of DF, FC. Q.E.D.

SCHOLIUM. When the two lines interſect each other in the centre of the circle, the rectangles of their ſegments will manifeſtly be equal, becauſe the ſegments themſelves are all equal.

PROP. XXVIII. THEOREM.

If two right lines be drawn from any point without a circle, to the oppoſite part of the circumference, the rectangle of the whole and external part of the one, will be equal to the rectangle of the whole and external part of the other.



Let ADFB be a circle, and AC, BC any two right lines, drawn from the point C, to the oppoſite part of the circumference; then will the rectangle of AC, CD be equal to the rectangle of BC, CF.

For, through the centre E, and the point C, draw the right line CH; and, from the point E, draw EG at right angles to AC (I. 12.) and join AE.

Then, ſince AEC is a triangle, and the perpendicular EG divides the chord AD into two equal parts (III. 3.), the line DC will be equal to the difference of the ſegments AG, GC.

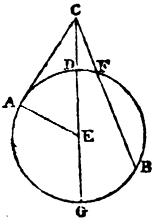
And, becauſe E is the centre of the circle, and AE is equal to EH, or EI, the line HC will be equal to the ſum of the ſides AE, EC, and IC will be equal to their difference.

But the rectangle under the ſum and difference of the two ſides of any triangle, is equal to the rectangle under the baſe and the difference of the ſegments of the baſe (*Cor.* II. 16.); whence the rectangle of HC, CI is equal to the rectangle of AC, CD.

And, in the ſame manner, it may be proved, that the rectangle of HC, CI is alſo equal to the rectangle of CB, CF: conſequently the rectangle of AC, CD will be equal to the rectangle of CB, CF. Q.E.D.

PROP. XXIX. THEOREM.

If two right lines be drawn from any point without a circle, the one to cut it, and the other to touch it; the rectangle of the whole and external part of the one, will be equal to the ſquare of the other.



Let CB, CA be any two right lines drawn from the point C, the one to cut the circle ADBG, and the other to touch it; then will the rectangle of CB, CF be equal to the ſquare of CA.

For, find E, the centre of the circle (III. 1.), and through the points E, C draw the line CEG; and join EA:

Then, ſince AC is a tangent to the circle, and EA is a line drawn from the centre to the point of contact, the angle CAE is a right angle (III. 12,)

And, becauſe EA is equal to EG, or ED, the line CG will be equal to the ſum of EA, EC, and CD will be equal to their difference.

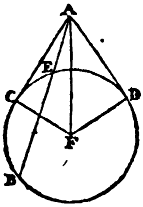
Since, therefore, the rectangle under the ſum and difference of any two lines is equal to the difference of their ſquares (II. 13.); the rectangle of CG, CD will be equal to the difference of the ſquares of CE, EA.

But the difference of the ſquares of CE, EA is equal to the ſquare of CA (*Cor.* II. 14.); therefore the rectangle of CG, CD will alſo be equal to the ſquare of CA.

And it has been ſhewn, in the laſt propoſition, that the rectangle of CG, CD is equal to the rectangle of CB, CF; conſequently the rectangle of CB, CF, will alſo be equal to the ſquare of CA. Q.E.D.

PROP. XXX. Theorem.

If two right lines be drawn from a point without a circle, the one to cut it, and the other to meet it; and if the rectangle of the whole and external part of the one be equal to the ſquare of the other, the latter will be a tangent to the circle.



Let AB, AC be two right lines, drawn from any point A, without the circle CBD, the one to cut it, and the other to meet it: then, if the rectangle of AB, AE be equal to the ſquare of AC, the line AC will be a tangent to the circle.

For, let F be the centre; and from the point A draw AD to touch the circle at D (III. 10.); alſo join FD FA, FC.

Then, ſince AD is a tangent to the circle, and AEB cuts it, the rectangle of AB, AE is equal to the ſquare of AD (III. 29.)

But the rectangle of AB, AE is alſo equal to the ſquare of AC (*by Hyp.*); whence the ſquare of AC is equal to the ſquare of AD, or AC equal to AD (II. 3.)

1 And,

And, becauſe FC is equal to FD, AC to AD, and AF common to each of the triangles AFC, AFD, the angle ACF will alſo be equal to the angle ADF (I. 21.)

But, ſince AD touches the circle, and DF is a line drawn from the centre to the point of contact, the angle ADF is a right angle (III. 12.)

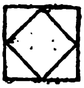
The angle ACF, therefore, is alſo a right angle; and CF produced is a diameter of the circle.

And ſince a right line, drawn from the end of the diameter, at right angles to it, touches the circle (III. 10.), AC will be a tangent to the circle CBD, as was to be ſhewn.

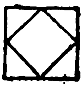
BOOK IV.

DEFINITIONS.

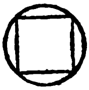
1. One rectilineal figure is ſaid to be inſcribed in another, when all the angles of the one are in the ſides of the other.



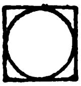
2. One rectilineal figure is ſaid to be deſcribed about another, when all the ſides of the one paſs through the angular points of the other.



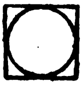
3. A rectilineal figure is ſaid to be inſcribed in a circle, when all its angular points are in the circumference of the circle.



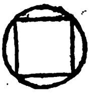
4. A rectilineal figure is ſaid to be deſcribed about a circle, when each ſide of it touches the circumference of the circle.



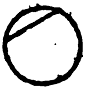
5. A circle is ſaid to be inſcribed in a rectilineal figure, when its circumference touches every ſide of that figure.



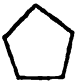
6. A circle is ſaid to be deſcribed about a rectilineal figure, when its circumference paſſes through all the angular points of that figure.



7. A right line is ſaid to be placed, or applied, in a circle, when the extremities of it are in the circumference of the circle.



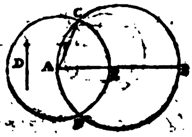
8. All plane figures contained under more than four ſides are called polygons; and if the angles, as well as ſides, are all equal, they are called regular polygons.



9. Polygons of five ſides, are called pentagons; thoſe of ſix ſides hexagons; thoſe of ſeven heptagons; and ſo on.

PROP. I. PROBLEM.

To place a right line in a given circle, equal to a given right line, not greater than the diameter of the circle.



Let ABC be a given circle, and D a given right line, not greater than the diameter; it is required to place a line in the circle ABC that ſhall be equal to D.

Find the centre of the circle (III. 1.), and draw any diameter AB; then if AB be equal to D the thing required is done.

But if not, make AE equal to D (I. 3.); and from the point A, at the diſtance AE, deſcribe the circle FEC, cutting the former in C.

Join the points A, C; and AC will be equal to D, as was required.

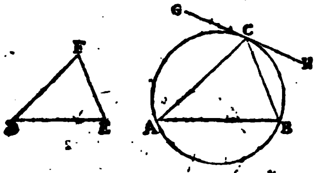
For ſince A is the centre of the circle CEF, AC is equal to AE.

But D is alſo equal to AE, by conſtruction; whence AC is, likewiſe, equal to D.

In the circle ABC, therefore, a right line has been placed equal to D, which was to be done.

PROP. II. PROBLEM.

To inſcribe a triangle in a given circle, that ſhall be equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle; it is required to inſcribe a triangle in the circle ABC, that ſhall be equiangular to the triangle DEF.

Draw the right line GH to touch the circle ABC in the point C (III. 10.); and, make the angle HCB equal to the angle D (I. 20.), and the angle GCA to the angle E; and join AB; then will ACB be the triangle required.

For, ſince the right line GH is a tangent to the circle, and CB is a chord drawn from the point of contact, the angle HCB will be equal to the angle CAB in the alternate ſegment (III. 24.)

But the angle HCB is equal to the angle D, by conſtruction; therefore the angle CAB is alſo equal to the angle D.

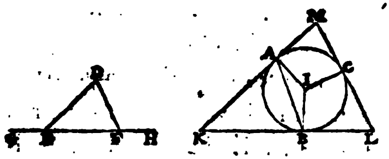
And, in the ſame manner, it may be proved, that the angle CBA is equal to the angle E.

But, ſince the angle CAB is equal to the angle D, and the angle CBA to the angle E, the remaining angle ACB will alſo be equal to the remaining angle F (*Cor.* I. 28.), and conſequently the triangle ACB is equianglar to the triangle DEF. Q. E. D.

PROP.

PROP. III. PROBLEM.

To circumſcribe a triangle about a given circle, that ſhall be equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle; it is required to circumſcribe a triangle about the circle ABC that ſhall be equiangular to the triangle DEF.

Produce the line EF to G and H; and, at the centre I, make the angles AIB, BIC equal to the angles DEG, DFH (I. 20.); and draw the lines MK, KL, LM, to touch the circle in the points A, B, C (III. 10.); and join AB.

Then, ſince the angles IAK, KBI, are, each of them, a right angle (III. 12.), the angles BAK, KBA, taken together, will be leſs than two right angles.

But when a right line interſects two other right lines, and makes the two interior angles, on the ſame ſide, together leſs than two right angles, thoſe lines will, if produced, meet each other (I. 25. *Cor.*)

The line MK, therefore, meets the line KL; and, if A, C, C, B be joined, the ſame may be proved of the lines KL, LM and MK; conſequently the figure KLM is a triangle.

And, becauſe the four angles of the quadrilateral AIBK are equal to four right angles (*Cor.* I. 28.), and the angles

IAK, KBI are each a right angle, the remaining angles AIB, BKA will be equal to two right angles.

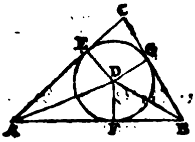
But the angles DEG, DEF are alſo equal to two right angles (I. 13.); therefore, ſince the angle DEG is equal to the angle AIB (*by Conſt.*), the remaining angle BKA will be equal to the remaining angle DEF.

And, in the ſame manner, it may be proved, that the angle CLB is equal to the angle DFE.

The angle MKL, therefore, being equal to the angle DEF, and the angle MLK to the angle DFE, the remaining angle KML will alſo be equal to the remaining angle EDF; and conſequently the triangle KLM is equiangular to the triangle EFD. Q. E. D.

PROP. IV. PROBLEM.

In a given triangle to inſcribe a circle.



Let ABC be the given triangle; it is required to inſcribe a circle in it.

Biſect the angles CAB, ABC, with the right lines AD, DB (I. 9.)

Then, ſince the angles DAB, DBA are leſs than two right angles (I. 28.), the lines AD, DB, will, if produced, meet each other (I. 25. *Cor.*)

And, if from the point of interſection D, there be drawn the perpendiculars DF, DG and DE, they will be the radii of the circle required.

For, ſince the angle EAD is equal to the angle DAF (*by Conſt.*), and the angle AED to the angle DFA, (being each of them right angles), the remaining angle EDA will alſo be equal to the remaining angle ADF (I. 28. *Cor.*)

The triangles ADE, DAF, therefore, being equiangular, and having the ſide AD common to both, the ſide DE will alſo be equal to the ſide DF (I. 21.)

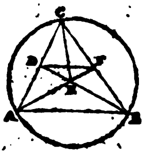
And, in the ſame manner, it may be proved, that the ſide DG is equal to the ſide DF.

The right lines DE, DG and DF are, therefore, all equal to each other; and the angles at the points F, E and G are right angles, by conſtruction.

If, therefore, a circle be deſcribed from the centre D, with either of the diſtances DE, DG or DF, it will touch the ſides in the points E, G, F (III. 10.) and be inſcribed in the triangle ABC, as was to be done.

PROP. V. PROBLEM.

To circumſcribe a circle about a given triangle.



Let ABC be the given triangle; it is required to circumſcribe a circle about it.

Biſect the ſides AC, CB with the perpendiculars DE, EB (I. 10 *and* 11.); and join DF.

Then, ſince the angles EDF, DFE are leſs than two right angles (*by Conſt.*), the lines DE, EF will meet each other (I. 25. *Cor.*)

Let E, therefore, be their point of interſection, and draw the lines EA, EC and EB.

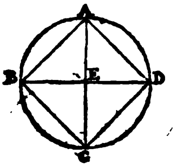
Then, becauſe AD is equal to DC (*by Conſt.*), DE common, and the angle ADE equal to the angle EDC (being each of them right angles), the baſe AE will alſo be equal to the baſe EC (I. 4.)

And, in the ſame manner, it may be proved, that EC is equal to EB; conſequently EA, EC and EB are all equal to each other.

If, therefore, a circle be deſcribed from the point E, at either of the diſtances EA, EC or EB, it will paſs through the remaining points, and circumſcribe the triangle ABC, as was to be done.

PROP. VI. PROBLEM.

To inſcribe a ſquare in a given circle.



Let ABCD be the given circle; it is required to inſcribe a ſquare in it.

Through E, the centre of the circle, draw any two diameters AC, BD at right angles to each other (I. 11, 12.), and join AB, BC, CD and DA; then will BCDA be the ſquare required.

For ſince the two ſides BE, EA, are equal to the two ſides ED, EA, and the angle BEA to the angle AED, (being each of them right angles), the baſe BA will be equal to the baſe AD (I. 4.)

And, in the ſame manner, it may be proved, that the ſides BC, CD are each equal to the ſides BA, AD; whence the figure BCDA is equilateral.

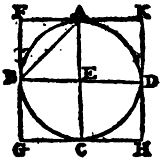
It is alſo rectangular: for, ſince BDA is a ſemi-circle, the angle BAD is a right angle (III. 12.)

And, for the ſame reaſon, the angles ABC, BCD and CDA are each of them right angles.

The figure BCDA, therefore, being equilateral, and having all its angles right angles is a ſquare, and it is inſcribed in the circle ABCD, as was to be done.

PROP. VII. PROBLEM.

To circumſcribe a ſquare about a given circle.



Let ABCD be the given circle; it is required to circumſcribe a ſquare about it.

Draw any two diameters AC, BD at right angles to each other (I. 11, 12.); and through the points A, B, C, D, draw the tangents KF, FG, GH, HK (III. 10.); and join AB.

Then, ſince the angles EAF, EBF are, each of them, right angles (III. 12.), the angles FAB, FBA will be, together, leſs than two right angles; whence the lines KF, FG will meet each other (I. 25. *Cor.*)

And, if the points A,D, D,C and C,B be joined, it may be proved, in like manner, that all the other lines FK, KH, HG and GF will meet each other.

And, ſince the angles at the points A, B, C, D are right angles (III. 12.), as alſo the angles at the point E (*by Conſt.*) the figure FH, and all the parts into which it is divided, will be parallelograms (I. 22, 23.)

But the oppoſite ſides of parallelograms are equal to each other (I. 30.); whence the ſides FG, GH, HK and KF, being each equal to the diameter AC, or BD, the figure FH will be equilateral.

It is, alſo, rectangular: for ſince FE is a parallelogram, and BEA is a right angle (*by Conſt.*), the angle F will, alſo, be a right angle (I. 28. *Cor.*)

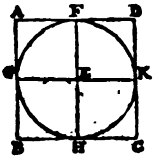
And, in the ſame manner, it may be proved, that the angles G, H and K are right angles.

The figure FH, therefore, being equilateral, and having all its angles right angles, is a ſquare; and it is circumſcribed about the circle ABCD, as was to be done.

PROP.

PROP. VIII. PROBLEM.

To inſcribe a circle in a given ſquare.



Let ABCD be the given ſquare; it is required to inſcribe a circle in it.

Biſect the ſides AD, AB in the points F and G (I. 10.), and draw FH, GK parallel to AB and AD (I. 27.); then will the point of interſection E be the centre of the circle required.

For, ſince AE is a parallelogram (*by Conſt.*), the ſide AF will be equal to the ſide GE, and the ſide AG to the ſide EF (I. 30.)

But the ſide AF is equal to the ſide AG, (being each of them equal to half the ſide of the ſquare AD or AB), whence the ſide GE will alſo be equal to the ſide EF.

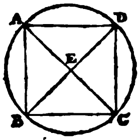
And, in the ſame manner, it may be proved, that HE, EK are equal to GE and EF.

The lines EF, EG, EH, and EK are, therefore, all equal to each other; and the angles at the points F, G, H and K are right angles, by the nature of parallel lines.

If, therefore, a circle be deſcribed from the point E, at the diſtance EF, EG, EH or EK, it will paſs through the remaining points, and be inſcribed in the ſquare AC, as was to be done.

PROP. IX.

To circumſcribe a circle about a given ſquare.



Let ABCD be the given ſquare; it is required to circumſcribe it with a circle.

Draw the diagonals AC, BD, and the point of interſection E will be the centre of the circle required.

For, ſince the ſides DA, AC are equal to the ſides BA, AC, and the baſe BC to the baſe CD, the angle DAC will be equal to the angle CAB (I. 7.): that is, the angle BAD will be biſected by the line AC.

And, in the ſame manner, it may be proved, that all the other angles of the ſquare are biſected by the lines DB and CA.

But the angles CDA, DAB, being right angles, are equal to each other; whence the angles EDA, EAD are alſo equal to each other; and conſequently the line ED is equal to the line EA (I. 7.)

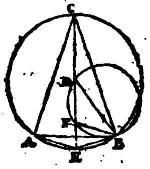
And, in like manner, it may be ſhewn, that the lines EB, EC are each equal to the lines ED, EA; whence the lines EA, EB, EC and ED are all equal to each other.

If, therefore, a circle be deſcribed from the point E, at either of the diſtances EA, EB, EC or ED, it will paſs through the remaining points, and circumſcribe the ſquare AC, as was to be done.

PROP.

PROP. X. PROBLEM.

To inſcribe an iſoſceles triangle in a given circle, that ſhall have each of the angles at its baſe double the angle at the vertex.



Let ABC be the given circle; it is required to inſcribe an iſoſceles triangle in it, that ſhall have each of the angles at its baſe double the angle at the vertex.

Draw any diameter CE, and divide the radius DE in the point F, ſo that the rectangle of DE, EF may be equal to the ſquare of FD (II. 22.)

From the point E apply the right lines EA, EB each equal to FD (IV. 1.), and join AB, AC, CB; then will ABC be the triangle required.

For, through the points D, F, B deſcribe the circle BDF (III. 18.), and draw the lines BD, BF.

Then, ſince the rectangle of DE, EF is equal to the ſquare of FD, or its equal EB, the line EB will touch the circle BDF, at the point B (III. 30.)

And, becauſe EB is a tangent to the circle, and BF is a chord drawn from the point of contact, the angle EBF will be equal to the angle FDB in the alternate ſegment (III. 24.)

And if, to each of theſe angles, there be added the angle FBD, the whole angle DBE or FEB will be equal to the angles FDB, FBD, taken together.

But the angle DBE is equal to the angle DEB, or FEB, (I. 5.), and the angles FDB, FBD to the angle EFB (I. 28.); whence the angle FEB will be equal to the angle EFB, and the ſide EB to the ſide BF (I. 5.)

And ſince EB is equal to FD, by conſtruction, BF will alſo be equal to FD, and the angle FDB to the angle FBD (I. 5.)

Theſe two angles, therefore, taken together, are double the angle FDB; whence the angle EFB, or its equal FEB, is alſo double the angle FDB.

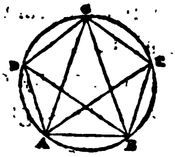
But the angle FEB, or CEB, is equal to the angle CAB (III. 15.), and the angle FDB, or EDB, to the angle ACB (III. 14. *and* I. *Ax.* 6.); conſequently the angle CAB is alſo double the angle ACB.

And, ſince EAC, EBC are right angled triangles (III. 16.), having EA equal to EB (*by Conſt.*) and EC common, the remaining ſide AC will be equal to the remaining ſide CB (III. 8. *Cor.*)

The triangle ABC, therefore, is iſoſceles, and has each of the angles at its baſe double the angle at the vertex; as was to be ſhewn.

PROP. XI. PROBLEM.

In a given circle to inſcribe a regular pentagon.



Let CDABE be the given circle; it is required to inſcribe a regular pentagon in it.

Make the iſoſceles triangle ABC ſuch, that each of the angles CAB, CBA may be double the angle ACB (IV. 10.)

Biſect the angles CAB, CBA with the lines AE, BD (I. 9.), and join the points AD, DC, CE, EB; then will ABECD be the pentagon required.

For, ſince the angles CAB, CBA are each double the angle ACB (*by Conſt.*), and the lines AE, BD biſect them, the angles ACB, CAE, EAB, ABD and DBC are all equal to each other.

And ſince equal angles ſtand upon equal circumferences (III. 21.), the arcs CD, DA, AB, BE and EC are alſo equal to each other.

But equal arcs are ſubtended by equal chords (III. 22.); conſequently the ſides CD, DA, AB, BE and EC are, likewiſe, equal.

The figure ABECD is, therefore, equilateral: and it is alſo equiangular.

For, ſince the arc CD is equal to the arc BE, to each of them add DAB, and the arc CDAB will be equal to the arc DABE.

But

For draw the lines OB, OA and OE, and let fall the perpendiculars OH, OK, OL, OF and OG (I. 12.):

Then, becauſe CB is equal to CD (*by Hyp.*), CO common, and the angle BCO equal to the angle OCD (*by Conſt.*), the angle CBO will alſo be equal to the angle ODC (I. 7.)

But the angle ODC is equal to half the angle CDE (*by Conſt.*), and the angle CDE is equal to the angle CBA (*by Hyp.*); conſequently the angle CBO is alſo equal to half the angle CBA.

The angle CBA, therefore, is biſected by the line BO; and, in the ſame manner, it may be ſhewn, that the angles at the points A, E are biſected, by the lines AO, OE.

Again, becauſe the triangles OGC, OCH are equiangular, and have OC common to each, the perpendicular OG will be equal to the perpendicular OH (I. 21.)

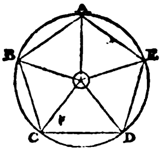
And, in the ſame manner, it may be ſhewn, that OH, OK, OL, OF and OG are all equal to each other.

If, therefore, a circle be deſcribed from the centre O, at either of theſe diſtances, it will paſs through the remaining points, and be inſcribed in the pentagon ABCDE, as was to be done.

PROP.

PROP. XIV. PROBLEM.

To deſcribe a circle about a given regular pentagon.



Let ABCDE be a given regular pentagon; it is required to circumſcribe it with a circle.

Biſect any two angles BCD, CDE, with the right lines CO, OD (I. 9.), and the point of interſection O will be the centre of the circle required.

For, draw the lines OB, OA and OE:

Then, becauſe CB is equal to CD (*by Hyp.*), CO common, and the angle BCO equal to the angle OCD (*by Conſt.*), the angle CBO will alſo be equal to the angle ODC (I. 4.)

But the angle ODC is equal to half the angle CDE, (*by Conſt.*), and the angle CDE is equal to the angle CBA (*by Hyp.*); whence the angle CBO is alſo equal to half the angle CBA.

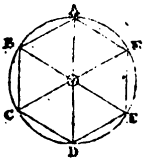
The angle CBA, therefore, is biſected by the line BO; and, in the ſame manner, it may be ſhewn, that the angles at the points A, E are biſected, by the lines AO, OE.

Since, therefore, the angle OCD is equal to the angle ODC (*by Hyp. and Ax.* 7.), the ſide OC will alſo be equal to the ſide OD (I. 5.)

And, in the ſame manner, it may be ſhewn, that OD, OE, OA, OB and OC are all equal to each other, If, therefore, a circle be deſcribed from the point O, at either of theſe diſtances, it will paſs through the remaining points, and circumſcribe the pentagon ABCDE, as was to be done.

PROP. XV. PROBLEM.

In a given circle to inſcribe a regular hexagon.



Let ACEF be a given circle; it is required to inſcribe a regular hexagon in it.

Through the centre O draw the diameter AD, and make DC equal to DO (IV. I.), and it will be the ſide of the hexagon required.

For, draw the diameter CF, and make BE parallel to CD [illegible] 27.); and join the points DE, EF, FA, AB and BC: Then, ſince DOC is an equilateral triangle, the angles ODC, OCD and DOC will be all equal to each other (I. 5. *Cor.*)

And, becauſe OE is parallel to CD, the angle EOD will be equal to the angle ODC (I. 24.), and the angle FOE to the angle OCD (I. 25.)

But the angles ODC, OCD are each equal to the angle DOC; therefore, the angles DOC, EOD and FOE are all equal to each other; as are alſo the oppoſite angles FOA, AOB and BOC.

Since, therefore, the triangles COD, DOE, &c. have two ſides, and the included angle of the one equal to two ſides and the included angle of the other, they will be equal in all reſpects (I. 4.)

The ſides CD, DE, EF, &c. are therefore all equal to each other, as are alſo the angles BCD, CDE, &c. whence ABCDEF is a regular hexagon; and it is inſcribed in the circle ACEF, as was to be done.

SCHOLIUM. Beſides the figures here conſtructed, and thoſe ariſing from thence by continual biſections, or taking the differences, no other regular polygon can be deſcribed, by any known method, purely geometrical.

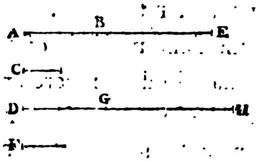
It may alſo be obſerved that ſome of theſe figures, as well as ſeveral others, in the former part of the work, may often be deſcribed in a much eaſier way, for practical purpoſes; but the principles upon which they depend can only be obtained from the following books of the Elements.

As many magnitudes, therefore, as there are in AB equal to E, ſo many are there in AB and CD together, equal to E and F together.

And, conſequently, whatever multiple AB is of E, the ſame multiple will AB and CD together be of E and F together. Q.E.D.

PROP. II. THEOREM.

If any number of magnitudes be multiples of the ſame magnitude, and as many others be the ſame multiples of another magnitude, each of each, the ſum of all the former will be the ſame multiple of the one, as the ſum of all the latter is of the other.



Let any number of magnitudes AB, BE, be multiples of the ſame magnitude C, and as many others DG, GH, the ſame multiples of another F, each of each; then will the whole AE, be the ſame multiple of C, as the whole DH, is of F.

For ſince AB is the ſame multiple of C that DG is of F (*by Hyp.*), there will be as many magnitudes in AB equal to C, as there are in DG equal to F.

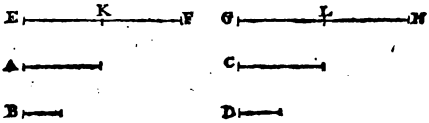
And becauſe BE is the ſame multiple of C that GH is of F (*by Hyp.*), there will be as many magnitudes in BE equal to C, as there are in GH equal to F.

As many magnitudes, therefore, as there are in the whole AE equal to C, ſo many will there be in the whole DH equal to F.

The whole AE, therefore, is the ſame multiple of C, as the whole DH is of F. Q.E.D.

PROP. III. THEOREM.

If the firſt of four magnitudes be the ſame multiple of the ſecond as the third is of the fourth; and if of the firſt and third there be taken equimultiples, theſe will alſo be equimultiples, the one of the ſecond, and the other of the fourth.



Let A the firſt, be the ſame multiple of B the ſecond, as C the third, is of D the fourth; and let EF and GH be equimultiples of A and C; then will EF be the ſame multiple of B, that GH is of D.

For ſince EF is the ſame multiple of A that GH is of C (*by Hyp.*), there will be as many magnitudes in EF equal to A, as there are in GH equal to C.

Divide EF into the magnitudes EK, KF each equal to A (I. 35.); and GH into the magnitudes GL, LH, each equal to C.

Then will the number of magnitudes EK, KF in the one, be equal to the number of magnitudes GL, LH in the other.

And becauſe A is the ſame multiple of B that C is of D (*by Hyp.*), and EK is equal to A, and GL to C (*by Conſt.*), EK will be the ſame multiple of B, that GL is of D.

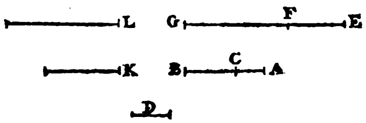
In like manner, ſince KF is equal to A, and LH to C, KF will be the ſame multiple of B, that LH is of D.

And ſince EK, KF are each multiples of B, and GL, LH are each the ſame multiples of D, the whole EF will be the ſame multiple of B, as the whole GH is of D (V. 2.)

Q.E.D.

PROP. IV. THEOREM.

If the firſt of three magnitudes be greater than the ſecond, and the third be any magnitude whatever, ſome equimultiples of the firſt and ſecond may be taken, and ſome multiple of the third ſuch, that the former ſhall be greater than that of the third, but the latter not greater.



Let AB, BC be two unequal magnitudes, and D any other magnitude whatever; then there may be taken ſome equimultiples of AB, BC, and ſome multiple of D ſuch, that the multiple of AB ſhall be greater than that of D, but the multiple of BC not greater.

For of BC, CA take any equimultiples GF, FE ſuch, that they may be each greater than D; and of D take the multiples K and L ſuch, that L may be that which is firſt greater than GF, and K that which is next leſs than L.

Then, becauſe L is that multiple of D which is the firſt that becomes greater than GF, the next preceding multiple K will not be greater than GF; that is GF will not be leſs than K.

And, ſince FE is the ſame multiple of AC that GF is of BC (*by Conſt.*), GF will alſo be the ſame multiple of BC that EG is of AB (V. 1.)

The magnitudes EG and GF are, therefore, equimultiples of the magnitudes AB and BC, and L is a multiple of D.

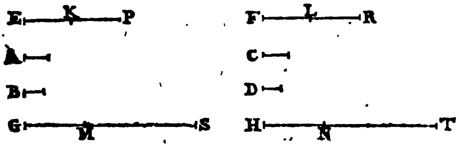
And, ſince GF is not leſs than K, and EF is greater than D (*by Conſt.*), the whole EG will be greater than K and D taken together.

But K and D, taken together, are equal to L (*by Conſt.*); therefore EG will be greater than L, and FG not greater than L, as was to be ſhewn.

PROP.

PROP. V. Theorem.

If four magnitudes be proportional, any equimultiples whatever of the antecedents will be proportional to any equimultiples whatever of the conſequents.



Let A be to B as C is to D, and of A and C take any equimultiples EK, FL; and of B and D any equimultiples GM, HN; then will EK be to GM, as FL is to HN.

For of EK and FL take any equimultiples whatever EP, FR; and of GM and HN any equimultiples whatever GS, HT:

Then, ſince EK is the ſame multiple of A, that FL is of C (*by Conſt.*), and of EK, FL have been taken the equimultiples EP, FR, EP will be the ſame multiple of A, that FR is of C (V. 3.)

And, in the ſame manner, it may be ſhewn, that GS is the ſame multiple of B, that HT is of D.

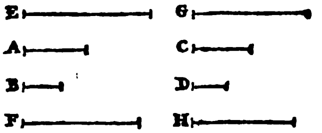
But A has the ſame ratio to B that C has to D (*by Hyp.*); and of A and C have been taken the equimultiples EP, FR; and of B and D the equimultiples GS, HT.

If, therefore, EP be greater than GS, FR will alſo be greater than HT; and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

And, ſince EP, FR are any equimultiples whatever of EK, FL; and GS, HT are any equimultiples whatever of GM, HN; EK will have the ſame ratio to GM, that FL has to HN (V. *Def.* 5.) Q.E.D.

PROP. VI. THEOREM.

If four magnitudes be proportional, and the firſt be greater than the ſecond, the third will alſo be greater than the fourth; and if equal, equal; and if leſs, leſs.



Let A have to B the ſame ratio that C has to D; then if A be greater than B, C will alſo be greater than D; and if equal, equal; and if leſs, leſs.

For, of A and C take any equimultiples E and G, and of B and D the ſame equimultiples F and H.

Then, becauſe A is to B, as C is to D (*by Hyp.*) if E be greater than F, G will alſo be greater than H; and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

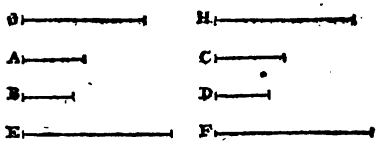
And, ſince E, F, G, H are the ſame multiples of A, B, C, D, each of each, theſe laſt magnitudes will alſo obſerve the ſame agreement of equality, exceſs, or defect with their equimultiples.

If, therefore, A be greater than B, C will alſo be greater than D; and if equal, equal; and if leſs, leſs. Q.E.D.

PROP.

PROP. VII. Theorem.

If four magnitudes be proportional, they will be proportional alſo when taken inverſely.



If A has to B the ſame ratio that C has to D; then, inverſely, B will have to A the ſame ratio that D has to C.

For, of B and D take any equimultiples whatever E and F; and of A and C any equimultiples whatever G and H:

Then, ſince A is to B as C is to D (*by Hyp.*), and G, H are equimultiples of A, C, and E, F of B, D (*by Conſt.*), if G be greater than E, H will be greater than F; and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

And, becauſe G has with E the ſame agreement of equality, exceſs, or defect, that H has with F, E will have with G the ſame agreement of equality, exceſs, or defect, that F has with H.

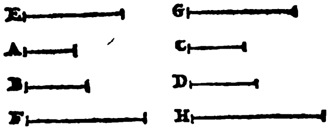
If, therefore, E be greater than G, F will alſo be greater than H; and if equal, equal; and if leſs, leſs.

But E and F are any equimultiples whatever of B and D (*by Conſt.*); and G and H are any equimultiples whatever of A and C; therefore, B is to A as D is to C (V. *Def.* 5.)

Q.E.D.

PROP. VIII. THEOREM.

If the firſt of four magnitudes be the ſame multiple or part of the ſecond as the third is of the fourth; the firſt will have the ſame ratio to the ſecond as the third has to the fourth.



Let A the firſt, be the ſame multiple of B the ſecond, that C the third is of D the fourth; then will A have to B the ſame ratio that C has to D.

For of A and C take any equimultiples whatever E and G; and of B and D any equimultiples whatever F and H:

Then, becauſe A is the ſame multiple of B that C is of D (*by Hyp.*), and E is the ſame multiple of A that G is of C (*by Conſt.*), E will alſo be the ſame multiple of B that G is of D (V. 3.)

And, ſince E is the ſame multiple of B that G is of D, and F is the ſame multiple of B that H is of D (*by Conſt.*), if E be greater than F, G will be greater than H; and if equal, equal; and if leſs, leſs.

But E and G are any equimultiples whatever of A and C; and F and H are any equimultiples whatever of B and D; therefore, A will have to B the ſame ratio that C has to D (V. *Def.* 5.)

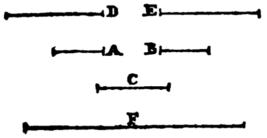
Again, let the firſt B, be the ſame part of the ſecond A, as the third D, is of the fourth C; then will B have to A the ſame ratio that D has to C.

For A is the ſame multiple of B that C is of D (*by Hyp.*); therefore A will have to B the ſame ratio that C has to D (V. 8.)

And, ſince A is to B as C is to D, therefore, alſo, inverſely, B is to A as D is to C (V. 7.) Q.E.D.

PROP. IX. THEOREM.

Equal magnitudes have the ſame ratio to the ſame magnitude, and the ſame has the ſame ratio to equal magnitudes.



Let A and B be equal magnitudes, and C any other magnitude whatever; then A will have to C the ſame ratio that B has to C.

For of A and B take any equimultiples whatever D and E; and of C any multiple whatever F:

Then, becauſe D is the ſame multiple of A, that E is of B, and A is equal to B, D will alſo be equal to E.

And, ſince D and E are equal to each other, if D be greater than F, E will alſo be greater than F; and if equal, equal; and if leſs, leſs.

But D and E are any equimultiples whatever of A and B, and F is any multiple whatever of C; therefore A is to C as B is to C (V. *Def.* 5.)

Again, let A and B be equal magnitudes, and C any other magnitude whatever; then C has to A the ſame ratio that it has to B.

For, having made the ſame conſtruction as before, D may, in like manner, be ſhewn to be equal to E:

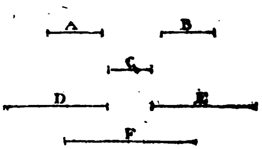
And, ſince D is equal to E, if F be greater than D, it will alſo be greater than E; or if equal, equal; or if leſs, leſs.

But F is any multiple whatever of C, and D and E are any equimultiples whatever of A and B; therefore, C is to A as C is to B (V. *Def.* 5.)

Q. E. D.

PROP. X. THEOREM.

Magnitudes which have the ſame ratio to the ſame magnitude are equal to each other; and thoſe to which the ſame magnitude has the ſame ratio are equal to each other.



Let A have to C the ſame ratio that B has to C; then will A be equal to B.

For, if they be not equal, one of them muſt be greater than the other.

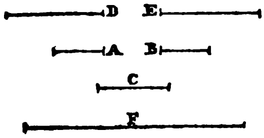
Again, let the firſt B, be the ſame part of the ſecond A, as the third D, is of the fourth C; then will B have to A the ſame ratio that D has to C.

For A is the ſame multiple of B that C is of D (*by Hyp.*); therefore A will have to B the ſame ratio that C has to D (V. 8.)

And, ſince A is to B as C is to D, therefore, alſo, inverſely, B is to A as D is to C (V. 7.) Q.E.D.

PROP. IX. THEOREM.

Equal magnitudes have the ſame ratio to the ſame magnitude, and the ſame has the ſame ratio to equal magnitudes.



Let A and B be equal magnitudes, and C any other magnitude whatever; then A will have to C the ſame ratio that B has to C.

For of A and B take any equimultiples whatever D and E; and of C any multiple whatever F:

Then, becauſe D is the ſame multiple of A, that E is of B, and A is equal to B, D will alſo be equal to E.

And, ſince D and E are equal to each other, if D be greater than F, E will alſo be greater than F; and if equal, equal; and if leſs, leſs.

But

Let A be the greater; and of A and B take the equimultiples D and E, and of C the multiple F ſuch, that D may be greater than F, and E not greater than F (V. 4.)

Then, ſince A is to C as B is to C, and D and E are equimultiples of A and B, and F is a multiple of C, D being greater than F, E will alſo be greater than F (V. *Def.* 5.)

But, by conſtruction, E is not greater than F; whence it is greater and not greater at the ſame time, which is abſurd.

The magnitude A is, therefore, not greater than B; and in the ſame manner it may be ſhewn that it is not leſs; conſequently they are equal to each other.

Again, let C have to A the ſame ratio that it has to B; then will A be equal to B.

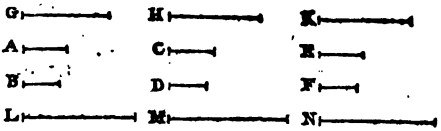
For, ſince C is to A as C is to B, therefore, alſo, inverſely, A will be to C as B is to C (V. 7.)

But magnitudes which have the ſame ratio to the ſame magnitude have been ſhewn to be equal to each other, therefore A is equal to B. Q. E. D.

PROP.

PROP. XI. THEOREM.

Ratios which are the ſame to the ſame ratio, are the ſame to each other.



Let A be to B as C is to D, and C to D as E is to F; then will A be to B as E is to F.

For, of A, C and E take any equimultiples whatever G, H and K; and of B, D and F any equimultiples whatever L, M and N:

Then, ſince A is to B as C is to D (*by Hyp.*), and G and H are equimultiples of A and C, and L and M of B and D, if G be greater than L, H will be greater than M; and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

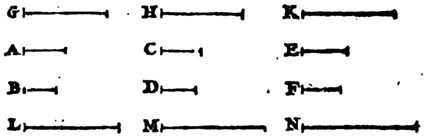
And, becauſe C is to D as E is to F (*by Hyp.*), and H and K are equimultiples of C and E, and M and N of D and F; if H be greater than M, K will be greater than N; and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

But if G be greater than L, it has been ſhewn that H will alſo be greater than M; and if equal, equal; and if leſs, leſs; whence, if G be greater than L, K will alſo be greater than N; and if equal, equal; and if leſs, leſs.

And ſince G and K are any equimultiples whatever of A and E, and L and N are any equimultiples whatever of B and F, A will have to B the ſame ratio that E has to F (V. *Def.* 5.) Q. E. D.

PROP. XII. THEOREM.

If any number of magnitudes be proportional, either of the antecedents will be to its conſequent, as the ſum of all the antecedents is to the ſum of all the conſequents.



Let A be to B as C is to D, and as E is to F; then will A be to B, as A, C and E together, are to B, D and F together.

For, of A, C and E take any equimultiples whatever G, H and K; and of B, D and F any equimultiples whatever L, M and N:

Then, ſince A is to B, as C is to D (*by Hyp.*), and G, H are equimultiples of A, C, and L, M of B, D, if G be greater than L, H will be greater than M, and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

And becauſe A is alſo to B as E is to F (*by Hyp.*), and G, K are equimultiples of A, E, and L, N of B, F, if G be greater than L, K will be greater than N; and if equal, equal; and if leſs, leſs (V. *Def.* V.)

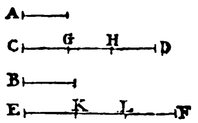
From hence it follows, that if G be greater than L, G, H and K together, will be greater than L, M and N together; and if equal, equal; and if leſs, leſs.

But G, and G, H, K together, are any equimultiples whatever of A, and A, C, E together (*by Conſt.*); and L, and L, M, N together, are any equimultiples whatever of B, and B, D, F together; whence, as A is to B, ſo is A, C and E together, to B, D and F together (V. *Def.* 5.)

Q. E. D.

PROP. XIII. THEOREM.

Equimultiples of any two magnitudes have the ſame ratio as the magnitudes themſelves.



Let CD be the ſame multiple of A that EF is of B; then will CD have the ſame ratio to EF that A has to B.

For, ſince CD is the ſame multiple of A that EF is of B: there are as many magnitudes in CD equal to A, as there are in EF equal to B.

Let CD be divided into the magnitudes CG, GH, HD, each equal to A (I. 25.); and EF into the magnitudes EK, KL, LF, each equal to B.

Then, the number of magnitudes CG, GH, HD in the one, will be equal to the number of magnitudes EK, KL, LF in the other.

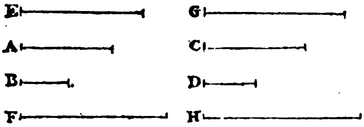
And, becauſe DH, HG, GC are all equal to each other, as are alſo FL, LK, KE, DH will be to FL as HG to LK, and as GC to KE (V. 9.)

And, ſince any antecedent is to its conſequent, as all the antecedents are to all the conſequents (V. 12.), FL will be to DH, as FE is to DC.

But DH is equal to A (*by Conſt.*), and FL is equal to B; therefore B will be to A, as FE to DC; and, inverſely, DC to FE as A to B. Q. E. D.

PROP. XIV. THEOREM.

If four magnitudes of the ſame kind be proportional, and the firſt be greater than the third, the ſecond will alſo be greater than the fourth; and if equal, equal; and if leſs, leſs.



Let A be to B as C is to D; then if A be greater than C, B will alſo be greater than D; and if equal, equal; and if leſs, leſs.

Firſt, let A be greater than C; then B will be greater than D.

For, of A, C take the equimultiples E, G; and of B the multiple F ſuch, that E may be greater than F, but G not greater (V. 4.); and make H the ſame multiple of D that F is of B.

Then, becauſe A is to B as C is to D (*by Hyp.*), and E, G are equimultiples of A, C, and F, H of B, D (*by Conſt.*), E being greater than F, G will alſo be greater than H (V. 5.)

And, ſince, by conſtruction, F is not leſs than G, and G has been proved to be greater than H, F will likewiſe be greater than H.

But F and H are equimultiples of B and D (*by Conſt.*); therefore, ſince F is greater than H, B will alſo be greater than D.

Secondly, let A be equal to C: then will B be equal D.

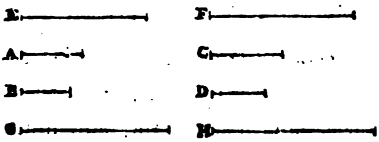
For, A is to B as C is to D (*by Hyp.*); or, ſince A is equal to C, A is to B as A is to D; therefore B is equal to D (V. 10.)

Thirdly, let A be leſs than C; then will B be leſs than D.

For, C is to D as A is to B, by the propoſition; therefore C being greater than A, D will alſo be greater than B, by the firſt caſe. Q. E. D.

PROP. XV. THEOREM.

If four magnitudes of the ſame kind be proportional, they will be proportional alſo when taken alternately.



Let A be to B as C is to D; then, alſo, alternately, A will be to C as B is to D.

For, of A and B take any equimultiples whatever E and G; and of C and D any equimultiples whatever F and H: Then,

Then, ſince E is the ſame multiple of A that G is of B, and that magnitudes have the ſame ratio as their equimultiples (V. 13.), A is to B as E is to G,

But A is to B as C is to D, by the propoſition; whence C is to D as E is to G (V. 11.)

In like manner, becauſe F is the ſame multiple of C that H is of D, C will be to D as F is to H (V. 13.)

But C has been ſhewn to be to D as E is to G; conſequently, E will be to G as F is to H (V. 11.)

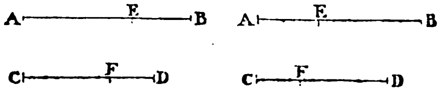
Since, therefore, E has the ſame ratio to G, that F has to H, if E be greater than F, G will alſo be greater than H; and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

But E and G are any equimultiples whatever of A and B; and F and H are any equimultiples whatever of C and D; therefore A is to C as B is to D (V. *Def.* 5.)

Q. E. D.

PROP. XVI. THEOREM.

If four magnitudes be proportional, the ſum of the firſt and ſecond, will be to the firſt or ſecond, as the ſum of the third and fourth, is to the third or fourth.



Let AE be to EB as CF is to FD; then will AB be to BE, or AE, as CD is to DF, or CF.

For, ſince AE is to EB as CF is to FD (*by Hyp.*); therefore, alternately, AE will be to CF as EB to FD (V. 15.)

And, ſince the antecedent is to its conſequent as all the antecedents are to all the conſequents (V. 12.), AE will be to CF as AB is to CD.

But ratios which are the ſame to the ſame ratio are the ſame to each other (V. 11.); whence AB will be to CD as EB is to FD; and, alternately, AB to EB as CD to DF (V. 15.)

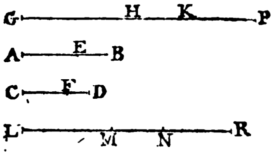
Again, ſince AE has been ſhewn to be to CF as AB is to CD, therefore, by alternation, AE will be to AB as CF is to CD (V. 15.)

But quantities which are directly proportional are alſo proportional when taken inverſely; whence AB will be to AE as CD is to CF (V. 7.)

Q. E. D.

PROP. XVII. THEOREM.

If four magnitudes be proportional, the difference of the firſt and ſecond, will be to the firſt or ſecond, as the difference of the third and fourth, is to the third or fourth.



Let AB be to BE as CD is to DF; then will AE be to AB, or EB, as CF is to CD, or FD.

For, of AE, EB, CF, FD take any equimultiples whatever GH, HK, LM, MN; and of EB, FD any other equimultiples whatever KP, NR:

Then, becauſe GH is the ſame multiple of AE that HK is of EB (*by Conſt.*), GH will be the ſame multiple of AE that GK is of AB (V. 1.)

But GH is the ſame multiple of AE that LM is of CF (*by Conſt.*); therefore GK is the ſame multiple of AB that LM is of CF.

In like manner, becauſe LM is the ſame multiple of CF that MN is of FD (*by Conſt.*) LM will be the ſame multiple of CF that LN is of CD (V. 1.)

But LM has been ſhewn to be the ſame multiple of CF that GK is of AB; therefore GK is the ſame multiple of AB that LN is of CD.

Again, becauſe HK, KP are the ſame multiples of EB, that MN, NR are of FD (*by Conſt.*), HP will be the ſame multiple of EB, that MR is of FD (V. 2.)

And, ſince AB is to BE as CD to DF (*by Hyp.*), and GK, LN are equimultiples of AB, CD, and HP, MR of BE, DF, if GK be greater than HP, LN will be greater than MR; and if equal, equal; and if leſs, leſs (V. *Def.* 5.)

From the two former of theſe take away the common part HK, and from the two latter, the common part MN; then if GH be greater than KP, LM will be greater than NR, and if equal, equal; and if leſs, leſs.

But GH, LM are any equimultiples whatever of AE, CF (*by Conſt.*), and KP, NR are any equimultiples whatever of EB, FD; whence AE is to CF as EB to FD (V. *Def.* 5.); and, alternately, AE to EB as CF to FD.

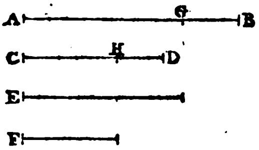
And, ſince AB is the ſum of AE, EB, and CD of CF, FD, AB will be to AE or EB, as CD is to CF or FD (V. 16.); and, inverſely, AE or EB to AB, as CF or FD to CD.

Q. E. D.

SCHOLIUM. When the conſequents are greater than the antecedents, the ſame demonſtration will hold, if the terms be taken inverſely.

PROP. XVIII. THEOREM.

If four magnitudes of the ſame kind be proportional, the greateſt and leaſt of them, taken together, will be greater than the other two.



Let AB, CD, E, F be the four proportional magnitudes, of which AB is the greateſt and F the leaſt; then will AB and F together, be greater than CD and E together.

For in AB take AG equal to E, and in CD take CH equal to F (I. 3.)

Then, becauſe AB is to CD as E to F (*by Hyp.*), and AG is equal to E and CH to F (*by Conſt.*), AB will be to CD as AG to CH.

But magnitudes which are proportional, are alſo proportional when taken alternately (V. 15.); therefore AB will be to AG as CD to CH.

And, ſince BG is the difference of AB and AG, and DH of CD and CH, BG will be to AB as DH to CD (V. 17.); and, inverſely, AB to BG as CD to DH.

But AB is greater than CD (*by Hyp.*); whence GB is alſo greater than HD (V. 14.)

And, becauſe AG is equal to E, and CH to F, AG and F together, are equal to CH and E together.

To the firſt of theſe equals add GB, and to the ſecond HD, then will AG, GB and F together, be greater than CH, HD and E together.

But AG, GB are equal to AB, and CH, HD to CD; conſequently AB and F together are greater than CD and E together. Q. E. D.

SCHOLIUM. That F muſt be the leaſt of the four magnitudes when A is the greateſt, appears from propoſitions VI. and XIII.

BOOK VI.

DEFINITIONS.

1. Similar rectilineal figures are thoſe which are equiangular, and have the ſides about the equal angles proportional.

2. The homologous, or like ſides of ſimilar figures are thoſe which are oppoſite to equal angles.

3. Two figures are ſaid to have their ſides reciprocally proportional, when the firſt conſequent and ſecond antecedent, of the four terms, are both ſides of the ſame figure.

4. Of three proportional quantities, the middle one is ſaid to be a mean proportional between the other two; and the laſt a third proportional to the firſt and ſecond.

5. Of four proportional quantities the laſt is ſaid to be a fourth proportional to the other three taken in order.

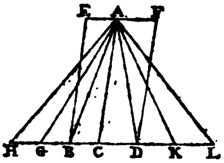
6. If any number of magnitudes be continually proportional, the ratio of the firſt and third is ſaid to be duplicate that of the firſt and ſecond; and the ratio of the firſt and fourth, triplicate that of the firſt and ſecond.

7. And of any number of magnitudes of the ſame kind, taken in order, the ratio of the firſt to the laſt, is ſaid to be compounded of the ratio of the firſt to the ſecond, of the ſecond to the third, and ſo on, to the laſt.

8. A right line is ſaid to be divided in extreme and mean ratio, when the whole line is to the greater ſeg[illegible]t, as the greater ſegment is to the leſs.

PROP. I.

Triangles and parallelograms, having the ſame altitude, are to each other as their baſes.



Let the triangles ABC, ACD, and the parallelograms EC, CF have the ſame altitude, or be between the ſame parallels BD, EF; then will the baſe BC be to the baſe CD, as the triangle ABC is to the triangle ACD, or as the parallelogram EC is to the parallelogram CF.

For, in BD produced, take any number of parts whatever BG, GH, each equal to BC; and DK, KL, any number whatever, each equal to CD; and join AG, AH, AK and AL:

Then, becauſe CB, BG, GH are all equal to each other, the triangles AHG, AGB, ABC will alſo be equal to each other (II. 5.); and whatever multiple the baſe HC is of the baſe BC, the ſame multiple will the triangle AHC be of the triangle ABC.

And, for the ſame reaſon, whatever multiple the baſe LC is of the baſe CD, the ſame multiple will the triangle ALC be of the triangle ADC.

If therefore, the baſe HC be equal to the baſe CL, the triangle AHC will be equal to the triangle ALC; and if greater, greater; and if leſs, leſs.

But the baſe HC, and the triangle AHC, are any equimultiples whatever of the baſe BC, and the triangle ABC; and the baſe CL and the triangle ALC are any equimultiples whatever of the baſe CD and the triangle ADC; whence the baſe BC is to the baſe CD, as the triangle ABC is to the triangle ACD (V. *Def.* 5.)

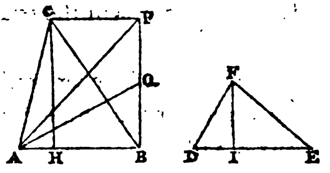
Again, becauſe the parallelogram CE is double the triangle ABC (I. 32.), and the parallelogram CF is double the triangle ADC, the triangle ABC will be to the triangle ADC as the parallelogram CE is to the parallelogram CF (V. 12.)

But, it has been ſhewn, that the baſe BC is to the baſe CD, as the triangle ABC is to the triangle ADC; therefore the baſe BC is alſo to the baſe CD, as the parallelogram CE is to the parallelogram CF. Q. E. D.

COROLL. Triangles and parallelograms, having equal altitudes, are to each other as their baſes.

PROP. II.

Triangles and parallelograms, having equal baſes, are to each other as their altitudes.



Let ABC, DEF be two triangles, having the equal baſes AB, DE, and whoſe altitudes are CH, FI; then will the triangle

And, ſince ratios which are the ſame to the ſame ratio, are the ſame to each other (V. 11.) DB will be to DA as EC is to EA; or, inverſely, AD to DB as AE to EC.

Again, let the ſides AB, AC be cut proportionally, in the points D and E; then will the line DE be parallel to BC.

For, the ſame conſtruction being made as before, the triangle DEB will be to the triangle DEA, as DB is to DA (VI. 1.); and the triangle EDC to the triangle DEA as EC to EA (VI. 1.)

And, ſince DB is to DA as EC is to EA (*by Conſt.*), the triangle DEB will be to the triangle DEA as the triangle EDC is to the triangle DEA (V. 11.)

But magnitudes which have the ſame ratio to the ſame magnitude are equal to each other (V. 10.); whence the triangle DEB is equal to the triangle EDC.

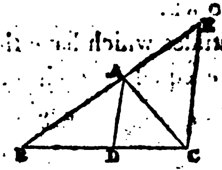
And ſince theſe triangles are equal to each other, and are upon the ſame baſe DE, they will have equal altitudes (VI. 2. *Cor.*), or ſtand between the ſame parallels; whence DE is parallel to BC.

Q.E.D.

COROLL. In the ſame manner it may be ſhewn, that the ſides of the triangle are proportional to any two of the parts into which they are divided; and that the like parts of each are alſo proportional.

PROP. IV. THEOREM.

If the verticle angle of a triangle be biſected, the ſegments of the baſe will have the ſame ratio with the other two ſides: and if the ſegments have the ſame ratio with the other two ſides, the angle will be biſected.



Let the angle BAC of the triangle ABC be biſected by the right line AD; then will BD be to DC as BA is to AC.

For through the point C draw CE parallel to DA (I. 27.), and let BA be produced to meet CE in E:

Then, becauſe the right line AC cuts the two parallel right lines AD, EC, the angle ACE will be equal to the alternate angle CAD (I. 24.)

But the angle CAD is equal to the angle BAD, by the propoſition; therefore the angle BAD is alſo equal to the angle ACE.

And, in like manner, becauſe the right line BE cuts the two parallel right lines AD, EC, the outward angle BAD will be equal to the inward oppoſite angle AEC (I. 25.)

But the angle ACE has been ſhewn to be equal to the angle BAD; whence the angle ACE is alſo equal to the angle AEC; and the ſide AE to the ſide AC (I. 5.)

And, ſince BEC is a triangle, and AD is drawn parallel to the ſide EC, BD will be to DC as BA is to AE (VI. 3.); &, becauſe AE is equal to AC, BD will be to DC as BA is to AC.

Again, let BD be to DC as BA is to AC; then will the angle BAC be biſected by the line AD.

For, let the ſame conſtruction be made as before:

Then ſince BD is to DC as BA is to AC, (*by Hyp.*), and BD to DC as BA to AE (VI. 3.), therefore, alſo, BA will be to AC as BA is to AE.

And ſince magnitudes which have the ſame ratio to the ſame magnitude are equal to each other (V. 10.), AC will be equal to AE, and the angle AEC to the angle ACE (I. 5.)

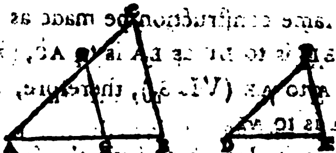
But the angle AEC is equal to the outward oppoſite angle BAD (I. 28.); and the angle ACE is equal to the alternate angle CAD (I. 29.); whence the angle BAD will alſo be equal to the angle CAD.

Q. E. D.

PROP.

PROP. V. THEOREM.

The ſides about the equal angles of equiangular triangles are proportional; and if the ſides about each of their angles be proportional, the triangles will be equiangular.



Let ABC, DEF be two equiangular triangles, of which BAC, EDF are correſponding angles; then will the ſide AB be to the ſide AC, as the ſide DE is to the ſide DF.

For make AG equal to DE, and AH to DF (I. 3.); and join GH.

Then, ſince the two ſides AG, AH of the triangle AHG, are equal to the two ſides DE, DF of the triangle DEF, and the angle A to the angle D, the angle AGH will alſo be equal to the angle DEF (I. 4.)

But the angle DEF is equal to the angle ABC (*by Hyp.*); conſequently the angle AGH will alſo be equal to the angle ABC, and GH will be parallel to BC (I. 28.)

And, ſince the line GH is parallel to the line BC, the ſide AB will be to the ſide AC, as the ſide AG is to the ſide AH (VI. 3.)

But AG is equal to DE, and AH to DF; whence the ſide AB will be to the ſide AC, as the ſide DE is to the ſide DF.

Again,

Again, let AB be to AC, as DE is to DF; and AB to BC, as DE to EF; then will the triangle ABC be equiangular with the triangle DEF.

For, let the ſame conſtruction be made as before:

Then, ſince AB is to AC as AG is to AH (*by Hyp.*), the line GH will be parallel to the line BC (VI. 3.), and the triangle AGH will be equiangular with the triangle ABC.

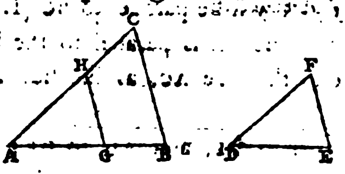
And, ſince the ſides about the equal angles of equiangular triangles are proportional (VI. 5.), the ſide AB will be to the ſide BC, as the ſide AG is to the ſide GH.

But the ſide AB is to the ſide BC, as the ſide DE is to the ſide EF (*by Hyp.*); therefore, alſo, the ſide AG will be to the ſide GH, as the ſide DE is to the ſide EF (V. 11.)

And, ſince the ſide AG, is equal to the ſide DE (*by Conſt.*) the ſide GH will alſo be equal to the ſide EF (V. 10.), and conſequently the triangle DEF will be equiangular with the triangle AGH (I. 7.) or ABC, as was to be ſhewn.

PROP. VI. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the ſides about the equal angles proportional, the triangles will be equiangular.



Let ABC, DEF be two triangles, having the angle A equal to the angle D, and the ſide AB to the ſide AC, as the ſide DE is to the ſide DF; then will the triangle ABC be equiangular with the triangle DEF.

For, make AG equal to DE, and AH to DF; and join GH:

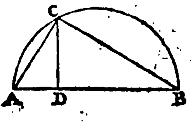
Then, ſince the ſides AG, AH are equal to the ſides DE, DF, and the angle A to the angle D (*by Hyp.*), the ſide GH will alſo be equal to the ſide EF, and the triangle AGH to the triangle DEF (I. 4.)

And, ſince AB is to AC as AG is to AH (*by Hyp.*), the line GH will be parallel to the line BC (VI. 3.); and conſequently the angle AGH is equal to the angle ABC, and the angle AHG to the angle ACB (I. 25.)

The triangle ABC is, therefore, equiangular with the triangle AGH, and conſequently it will alſo be equiangular with the triangle DEF. Q. E. D.

PROP. VII. THEOREM.

In a right angled triangle, a perpendicular from the right angle, is a mean proportional between the ſegments of the hypotenuſe; and each of the ſides is a mean proportional between the adjacent ſegment and the hypotenuſe.



Let ABC be a right angled triangle, and CD a perpendicular from the right angle to the hypotenuſe; then will

CD be a mean proportional between AD and DB; AC a mean proportional between AB and AD; and BC between AB and BD.

For, ſince the angle BDC is equal to the angle ACB (I. 8.), and the angle B is common, the triangles DBC, ABC will be equiangular (I. 28, *Cor.*)

And, in the ſame manner, it may be ſhewn, that the triangles ADC, ABC are equiangular; whence the triangle ADC is alſo equiangular with the triangle DBC.

But the ſides of equiangular triangles are proportional (VI, 5.); therefore the ſide AD is to the ſide CD, as the ſide CD is to the ſide DB.

In like manner, ſince the triangles ADC, DBC are each of them equiangular with the triangle ABC; AB will be to AC as AC is to AD; and AB to BC as BC is to BD (VI. 5.).

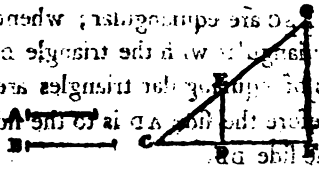
Q.E.D.

COROLL. If ACB be a ſemicircle, and CD a perpendicular let fall from any point C; then will AD × DB = DC^2, AB × AD = AC^2, and AB × BD = BC^2.

PROP.

PROP. VIII. PROBLEM.

To find a third proportional to two given right lines.



Let A, B be two given right lines; it is required to find a third proportional to them.

Draw the two indefinite right lines CF, CG, making any angle C, and take CD equal to A, and CE, DF each equal to B (I. 3.)

Also join the points D, E, and make FG parallel to DE (I. 27.); and EG will be the third proportional required.

For, ſince CFG is a triangle, and DE is parallel to FG (*by Conſt.*) CD will be to DF as CE is to EG (VI. 2.)

But DF is equal to CE, by conſtruction; therefore CD is to CE as CE is to EG.

And, ſince CD is equal to A, and CE to B (*by Conſt.*), A will be to B as B is to EG.

Q. E. D.

SCHOLIUM. A third proportional to two given right lines, may alſo be found by means of the laſt propoſition.

For let AD, DC be two given right lines, one of which DC is perpendicular to the other.

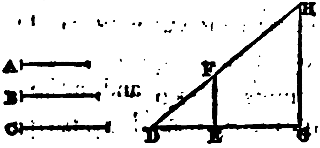
Then if CB be drawn at right angles to AC, and AD be produced till it meets the former in B, DB will be a third proportional to AD and DC as was required.

Again, let BD, DC be the two given right lines, placed at right angles to each other, as before;

Then, if CA be drawn at right angles to BC, and BD be produced till it meets the former in A, DA will be a third proportional to BD and DC.

PROP. IX. PROBLEM.

To find a fourth proportional to three given right lines.



Let A, B, C be three given right lines; it is required to find a fourth proportional to them.

Draw the two indefinite right lines DG, DH, making any angle D; and take DE equal to A, DF to B, and EG to C (I. 3.)

Join the points E, F, and make GH parallel to EF (I. 27.); and FH will be the fourth proportional required.

For, since DGH is a triangle, and EF is parallel to GH (*by Conſt.*) DE will be to DF as EG is to FH (VI. 8.)

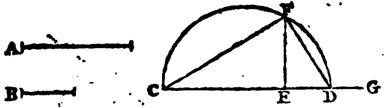
But DE is equal to A, DF to B, and EG to C (*by Conſt.*); therefore A will be to B as C is to FH (V. 9.)

Q.E.D.

PROP.

PROP. X. PROBLEM.

To find a mean proportional between two given right lines.



Let A, B be two given right lines; it is required to find a mean proportional between them.

Draw the indefinite right line CG, in which take CE equal to A, and ED equal to B (I. 3.)

Upon CD deſcribe the ſemi-circle CFD, and from the point E erect the perpendicular EF (I. 11.); and it will be the mean proportional required.

For join the points CF, FD:

Then, becauſe DFC is a ſemi-circle, the angle CFD is a right angle (III. 16.), and conſequently the triangle CDF is rectangular.

And ſince a perpendicular, from the right angle is a mean proportional between the ſegments of the hypotenuſe (VI. 7.), EF will be a mean proportional to CE, ED.

But CE is equal to A, and ED to B (*by Conſt.*); therefore EF will be a mean proportional to A and B, as was to be ſhewn.

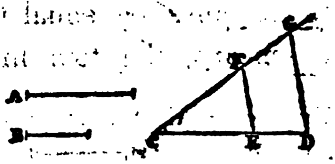
SCHOLIUM. If CD, DE be the two given lines, DF will be a mean proportional to them.

And if CD, CE be the given lines, CF will be a mean proportional to them (VI. 7.)

PROP.

PROP. XI. PROBLEM.

To divide a given right line into two parts which ſhall have the ſame ratio with two given right lines.



Let CD be a given right line, and A, B two other given right lines; it is required to divide CD into two parts which ſhall be to each other in the ratio of A to B.

Draw the indefinite right line CG, making any angle with CD; and make CF equal to A, and FG to B (I. 3.)

Join the points GD; and make FE parallel to GD (I. 27.); and it will divide CD in the point E as was required.

For, ſince CDG is a triangle, and FE is parallel to GD (*by Conſt.*), CE will be to ED as CF is to FG (VI. 3.)

But CF is equal to A, and FG to B (*by Conſt.*); therefore CE will be to ED as A is to B (V. 9.)

Q. E. D.

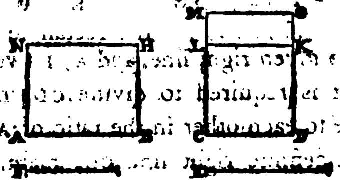
SCHOLIUM. In nearly the ſame manner may a third, fourth, or any other part be cut off from a given line CD.

For if CG be made the ſame multiple of CF that CD is of the part, and GD, FE be drawn as above, CE will be the part required.

PROP.

PROP. XII. THEOREM.

If four right lines be proportional, the rectangle of the extremes will be equal to the rectangle of the means; and if the rectangle of the extremes be equal to the rectangle of the means, the four lines will be proportional.



Let AB be to CD as E is to F; then will the rectangle of AB and F, be equal to the rectangle of CD and E.

For make BH perpendicular to AB, and equal to F (I. 10, 3.), and DG perpendicular to CD, and equal to E, and in DG take DK equal to BH (I. 3.) and draw KL parallel to CD (I. 27.)

Then, ſince parallelograms of equal altitudes, are to each other as their baſes (VI. 1.) the parallelogram AH will be to the parallelogram CK, as AB is to CD.

And ſince AB is to CD as E is to F (*by Hyp.*), or as DG is to DK (*by Conſt.*), the parallelogram AH will alſo be to the parallelogram CK as DG is to DK (V. 11.)

But parallelograms of the ſame baſe, are to each other as their altitudes (VI. 2.); whence the parallelogram CG will alſo be to the parallelogram CK as DG is to DK.

And becauſe ratios which are the ſame to the ſame ratio are the ſame to each other (V. 11.), the parallelogram AH will be to the parallelogram CK, as the parallelogram CG is to the parallelogram CK.

But magnitudes which have the ſame ratio to the ſame magnitude are equal to each other (V. 10.); whence the rectangle AH is equal to the rectangle CG.

Again, let AH, the rectangle of the extremes, be equal to CG, the rectangle of the means; then will AB be to CD as E is to F.

For, let the ſame conſtruction be made as before:

Then, ſince rectangles of equal altitudes are to each other as their baſes (VI. 2.), the rectangle AH will be to the rectangle CK as AB is to CD.

And, becauſe the rectangle AH is equal to the rectangle CG (*by Hyp.*), CG will alſo be to CK as AB is to CD (V. 9.)

But CG is to CK as DG is to DK or BH (VI. 2.), conſequently AB will be to CD as DG is to BH. (V. 11.)

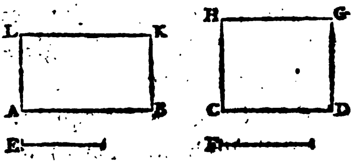
And ſince DG is equal to E, and BH to F, AB will be to CD as E is F (V. 9.)

Q. E. D.

PROP.

PROP. XIII. THEOREM.

If three right lines be proportional, the rectangle of the extremes will be equal to the ſquare of the mean: and if the rectangle of the extremes be equal to the ſquare of the mean, the three lines will be proportional.



Let AB be to CD as CD is to E; then will the rectangle of AB and E be equal to the ſquare of CD.

For make BK perpendicular to AB (I. 10.) and equal to E (I. 3.) and upon CD deſcribe the ſquare CG (II. 1.), and make F equal to CD.

Then, ſince CD is equal to F, and AB is to CD as CD is to E (*by Hyp.*), AB will alſo be to CD as F is to E (V. 9.)

And, ſince theſe four lines are proportional, the rectangle of AB and E will be equal to the rectangle of CD and F (VI. 12.)

But the rectangle of CD and F is equal to the ſquare of CD, becauſe CD is equal to F; therefore, alſo, the rectangle of AB and E will be equal to the ſquare of CD.

Again, if the rectangle of AB and E be equal to the ſquare of CD; AB will be to CD as CD is to E.

For let the ſame conſtruction be made as before:

Then, ſince the rectangle of AB and E is equal to the ſquare of CD (*by Hyp.*), and the ſquare of CD is equal to the rectangle of CD and F (II. 2.), the rectangle of AB and E will alſo be equal to the rectangle of CD and F.

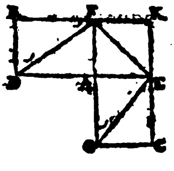
But if the rectangle of the extremes be equal to that of the means, the four lines are proportional (VI. 12.); whence AB is to CD as F is to E.

And ſince CD is equal to F, by conſtruction, AB will be to CD as CD is to E.

Q. E. D.

PROP. XIV. THEOREM.

Equal parallelograms and triangles have their ſides about equal angles reciprocally proportional; and if the ſides about equal angles are reciprocally proportional, the parallelograms and triangles will be equal.



Let AB, AC be two equal parallelograms, and DFA AEG two equal triangles, having the angle DAF equal to the angle GAE; then will the ſide DA be to the ſide AE, as the ſide AG is to the ſide AF.

For let the ſides DA, AE be placed in the ſame right line, and complete the parallelogram AK.

Then, becauſe the angles DAF, FAE are equal to two right angles (I. 15.), and the angle FAE is equal to the angle DAG (I. 13.), the angles DAF, DAG are alſo equal to two right angles; and conſequently FAG is a right line.

And ſince the parallelogram AB is equal to the parallelogram AC (*by Hyp.*), and AK is another parallelogram, AB is to AK as AC is to AK (V. 9.)

But AK is to AC as DA to AE (VI. 1.), and AK to AC as AG to AF; whence DA is to AE as AG is to AF (V. 11.)

And, if FE be joined, it may be ſhewn, in like manner, that the triangle DAF is to the triangle AFE as the triangle GAE is to the triangle AFE; and DA to AE as AG to AF.

Again, let the angle DAF be equal to the angle GAE, and the ſide BA to the ſide AE as the ſide AG is to the ſide AF; then will the parallelogram AB be equal to the parallelogram AC, and the triangle DFA to the triangle GAE.

For ſince DA is to AE as AG to AF (*by Hyp.*), and DA to AE as AB to AK (VI. 1.), AG will be to AF as AB to AK (V. 11.)

But AG is to AF as AC to AK (VI. 1.); whence AB is to AK as AC to AK (V. 11.); and conſequently the parallelogram AB is equal to the parallelogram AC (V. 10.)

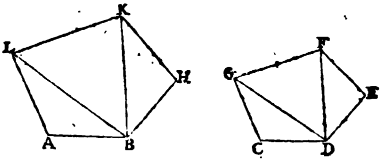
And ſince triangles are the halves of parallelograms, which have the ſame baſe and altitude, the triangle DFA will be equal to the triangle GAE.

Q. E. D.

COROLL. The ſides of equal rectangles are reciprocally proportional; and if the ſides are reciprocally proportional, the rectangles will be equal.

PROP. XV. PROBLEM.

Upon a given right line to deſcribe a rectilineal figure, ſimilar, and ſimilarly ſituated, to a given rectilineal figure.



Let AB be the given right line, and CDEFG the given rectilineal figure; it is required to deſcribe a rectilineal figure upon AB, which ſhall be ſimilar, and ſimilarly ſituated to CDEFG.

Join DG, DF; and at the points A, B, make the angles BAL, ABL equal to the angles DCG, CDG (I. 20.)

In like manner, at the points B, L, make the angles BLK, LBK equal to the angles DGF, GDF: and BKH, KBH equal to DFE, FDE.

Then, becauſe two angles in one triangle are equal to two angles in another, each to each, the remaining angles in each of the correſponding triangles, will alſo be equal (I. 28. *Cor.*)

And ſince the angles ALB, BLK, are equal to the angles CGD, DGF, and LKB, BKH to GFD, DFE, the angle ALK will be equal to the angle CGF, and the angle LKH to the angle GFE.

And, in the ſame manner, it may be ſhewn that the angles KHB, HBA and BAL are equal to the angles FED, EDC and DCG.

The figures ABHKL and CDEFG are, therefore, equiangular: and they have their ſides about the equal angles, alſo, proportioned.

For, ſince the triangles ALB, CGD are equiangular, AL will be to LB as CG to GD (VI. 5.); or AL to CG as LB to GD (V. 15.)

And, in like manner, LK will be to LB as GF to GD (VI. 5.); or LK to GF as LB to GD (V. 15.)

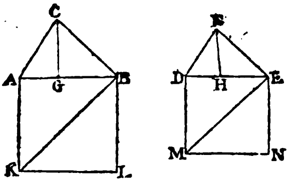
But ratios which are the ſame to the ſame ratio are the ſame to each other (V. 11.); whence AL will be to CG as LK to GF; or AL to LK as CG to GF (V. 15.)

And, in the ſame manner, it may be ſhewn, that the ſides about the angles K, H, B, A, are proportional to the ſides about the angles F, E, D, C.

The figure ABHKL is, therefore, ſimilar, and ſimilarly ſituated with the figure CDEFG (VI. *Def.* 1.); and it is deſcribed upon the right line AB, as was to be done.

PROP. XVI. THEOREM.

Equiangular, or ſimilar triangles, are to each other as the ſquares of their homologous ſides.



Let ABC, DEF be two ſimilar triangles, of which the ſides AB, DE are homologous; then will the triangle ABC be to the triangle DEF as the ſquare of AB is to the ſquare of DE.

For, on AB, DE deſcribe the ſquares AL, DN (II. 1.), and let fall the perpendiculars CG, FH (I. 12.)

Then, ſince the triangles ABC, DEF are ſimilar (*by Hyp.*), AC will be to AB as DF to DE (VI. *Def.* 1;); or AC to DF as AB to DE (V. 15.)

And, becauſe the triangles AGC, DHF are equiangular, AC will be to CG as DF to FH (VI. 5.); or AC to DF as CG to FH (V. 15.)

But ratios which are the ſame to the ſame ratio, are the ſame to each other (V. 11.); therefore CG is to FH as AB to DE; or CG to AB as FH to DE (V. 15.)

And ſince triangles which have the ſame baſe, are to each other as their altitudes (VI. 2.), the triangle ABC is to the triangle AKB as CG is to AK, or AB.

In the ſame manner it may be ſhewn, that the triangle DEF is to the triangle DME as FH is to DM, or DE.

But CG has been ſhewn to be to AB as FH is to DE; therefore the triangle ABC is to the triangle AKB as the triangle DEF is to the triangle DME (V. 11.)

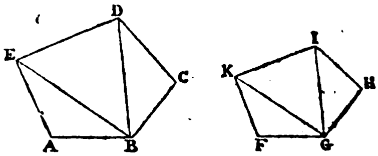
And ſince the ſquare AL is double the triangle AKB (I. 32.), and the ſquare DN is double the triangle DME, the triangle ABC will be to the triangle DEF as the ſquare AL is to the ſquare DN (V. 13 *and* 15.)

Q. E. D.

PROP.

PROP. XVII. THEOREM.

Similar polygons are to each other as the ſquares of their homologous ſides.



Let ABCDE, FGHIK be ſimilar polygons, of which AB, FG are homologous ſides; then will the polygon ABCDE be to the polygon FGHIK as the ſquare of AB is to the ſquare of FG.

For join the points BE, BD, GK and GI:

Then, ſince the angle A is equal to the angle F, and AB is to AE as FG is to FK (VI. *Def.* 1.), the triangles EAB, KFG will be equiangular, or ſimilar (VI. 6.)

And if, from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, the remaining angles BED, GKI will alſo be equal to each other.

But ED is to KI as EA is to KF (VI. *Def.* 1. *and* V. 15.), and EA is to KF as EB to KG (VI. 5. *and* V. 15.); whence ED will be to KI as EB is to KG (V. 11.)

Since, therefore, the angles BED, GKI are equal to each other, and the ſides about them are proportional, the triangles BED, GKI will, alſo, be equiangular, or ſimilar (VI. 6.)

And, in the ſame manner, it may be ſhewn, that the triangles BCD, GHI are equiangular, or ſimilar.

But ſimilar triangles are as the ſquares of their like ſides (VI. 16.); whence the triangle EAB is to the triangle KFG as the ſquare of EB is to the ſquare of KG.

And, for the ſame reaſon, the triangle EBD is to the triangle KGI as the ſquare of EB is to the ſquare of KG.

But ratios which are the ſame to the ſame ratio, are the ſame to each other (V. 11.); whence the triangle EAB is to the triangle KFG as the triangle EBD is to the triangle KGI.

And in the ſame manner it may be ſhewn that the triangle EBD is to the triangle KGI as the triangle DBC is to the triangle IGH.

The triangle EAB, therefore, is to the triangle KFG, as the triangle EBD is to the triangle KGI, and as the triangle DBC is to the triangle IGH (V. 11.)

And ſince the ſum of the antecedents is to the ſum of the conſequents as the firſt antecedent is to its conſequent (V. 16.), the polygon ABCDE will be to the polygon FGHIK as the triangle EAB is to the triangle KFG.

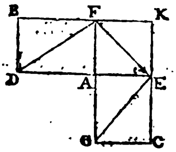
But the triangle EAB is to the triangle KFG as the ſquare of AB is to the ſquare of FG (VI. 16.); whence the polygon ABCDE is alſo to the polygon FGHIK as the ſquare of AB is to the ſquare of FG.

Q.E.D.

PROP.

PROP. VIII. THEOREM.

Parallelograms and triangles, having two equal angles, are to each other as the rectangles of the ſides which are about thoſe angles.



Let AB, AC be two parallelograms, having the angle DAF equal to the angle GAE; then will AB be to AC as the rectangle of DA, AF is to the rectangle of GA, AE.

For let the ſides DA, AE be placed in the ſame right line, and complete the parallelogram AK.

Then, becauſe the angles DAF, FAE, are equal to two right angles (I. 13.), and the angle FAE is equal to DAG (I. 15.), the angles DAF, DAG are alſo equal to two right angles; whence FG is a right line (I. 14.)

And ſince parallelograms, of the ſame altitude, are to each other as their baſes (VI. 1.), the parallelogram AB is to the parallelogram AK as AD is to AE.

But AD is to AE as the rectangle of AD, AF is to the rectangle of AE, AF (VI. 2. *Cor.* 2.); therefore AB is to AK as the rectangle of AD, AF is to the rectangle of AE, AF (V. 11.)

And in the ſame manner it may be ſhewn, that AC is to AK as the rectangle of AG, AE is to the rectangle of

AE, AF; whence AB is to AC as the rectangle of AD, AF is to the rectangle of AG, AE (V. 11 *and* 15.)

Again, let DFA, AEG be two triangles, having the angle DAF equal to the angle GAE, then will DFA be to AEG as the rectangle of DA, AF is to the rectangle of GA, AE.

For let the ſides DA, AE be placed in the ſame right line; and complete the parallelograms AB, AC, AK.

Then, as before, AB is to AC as the rectangle of DA, AF is to the rectangle of AG, AE.

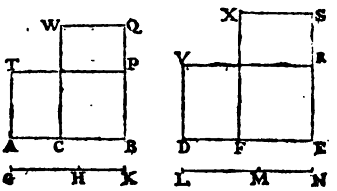
But the triangles DFA, AEG are half the parallelograms AB, AC (I. 30.); whence DFA is to AEG as the rectangle of DA, AF is to the rectangle of GA, AE.

Q. E. D.

SCHOLIUM. If the line EF be drawn, the latter part of this propoſition may be proved from the triangles, independently of the former.

PROP. XIX. THEOREM.

The rectangles under the correſponding lines of two ranks of proportionals, are themſelves proportionals.



Let AB be to CB as DE is to FE, and GH to GK as LM to LN; then will the rectangle of AB, GH be to that of

CB, GK as the rectangle of DE, LM is to that of FE, LN.

For draw BQ, ES at right angles to AB, DE (I. 11.), and make BP equal to GH, BQ to GK, ER to LM, and ES to LN (I. 3.): and complete the parallelograms AP, CQ, DR and FS.

Then ſince parallelograms, of the ſame altitude, are to each other as their baſes (VI. I.), AP will be to CP as AB to CB; and DR to FR as DE to FE.

But AB is to CB as DE to FE (*by Hyp.*); whence AP will be to CP as DR to FR (V. 11.), or AP to DR as CP to FR (V. 15.)

And ſince parallelograms of the ſame baſe are to each other as their altitudes (VI. 2.), CP will be to CQ as BP to BQ; and FR to FS as ER to ES.

Or, becauſe BP, BQ are equal to GH, GK, and ER, ES to LM, LN (*by Conſt.*), CP will be to CQ as GH to GK; and FR to FS as LM to LN (V. 9.)

But GH is to GK as LM to LN (*by Hyp.*), therefore CP will be to CQ as FR is to FS (V. 11.), or CP to FR as CQ to FS (V. 15.)

And it has been before ſhewn that AP is to DR as CP to FR; whence AP is to DR as CQ to FS (V. 11.), or AP to CQ as DR to FS (V. 15.)

But AP is the rectangle of AB, GH; CQ of CB, GK; DR of DE, LM; and FS of FE, LN (*by Conſt.*); therefore the rectangle of AB, GH is to that of CB, GK as the rectangle of DE, LM is to that of FE, LN.

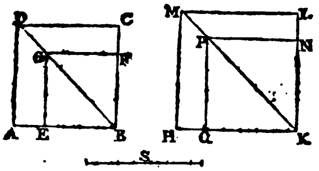
Q. E D.

COROLL. The ſquares of four proportional lines are themſelves proportionals.

PROP.

PROP. XX. THEOREM.

The ſides and diagonals of four proportional ſquares, are themſelves proportional.



Let AC, EF, HL and QN be four proportional ſquares; then will their ſides and diagonals be alſo proportionals.

For, make S a fourth proportional to AB, DB and HK (VI. 9.); and draw the diagonals BD and KM.

Then, ſince AB is to EB as HK is to S (*by Conſt.*), AC will be to EF as HL is to the ſquare of S (VI. 19. *Cor.*)

And becauſe AC is to EF as HL is to QN (*by Hyp.*), HL will be to the ſquare of S as HL is to QN, or the ſquare of QK (V. 11.)

But magnitudes which have the ſame ratio to the ſame magnitude are equal to each other (V. 10.); whence the ſquare of S is equal to the ſquare of QK.

And ſince equal ſquares have equal ſides (II. 8.), S is equal to QK; and conſequently AB is to EB as HK to QK.

Again, becauſe the triangles ABD, EBG are equiangular, AB will be to EB as BD to BG (VI. 5.)

And becauſe the triangles HKM, QKP are alſo equiangular, HK will be to QK, as KM to KP (VI. 5.)

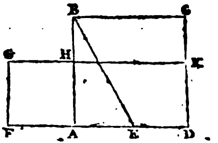
But AB has been ſhewn to be to EB as HK is to QK; conſequently BD is to BG as KM is to KP (V. 11.)

Q. E. D.

PROP.

PROP. XXI. PROBLEM.

To cut a given right line in extreme and mean proportion.



Let AB be the given right line; it is required to cut it in extreme and mean proportion.

Upon AB deſcribe the ſquare AC (II. 1.), and biſect the ſide AD in E (I. 10.) and join BE.

In EA produced, take EF equal to EB (I. 3.); and upon AF deſcribe the ſquare FH (II. 1.); then will AB be divided at the point H as was required.

For ſince DF is the ſum of EB, ED, or EB, EA, and AF is their difference, the rectangle of DF, FA is equal to the difference of the ſquares of EB, EA (II. 13.)

But the rectangle of DF, FA is equal to DG, becauſe FA is equal to FG; and the difference of the ſquares of EB, EA is equal to the ſquare of AB (II. 14. *Cor.*); whence DG is equal to AC.

And if from each of theſe equals, the part AK, which is common, be taken away, the remainder AG will be equal to the remainder HC.

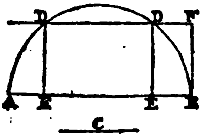
But equal parallelograms have the ſides about equal angles reciprocally proportional (VI. 15.); whence HK is to HG as HA to HB.

And ſince HK is equal to AD, or AB, and HG to HA, AB will be to HA as HA is to HB. Q.E.D.

PROP.

PROP. XXII. PROBLEM.

To divide a given right line into two ſuch parts, that their rectangle may be equal to a given ſquare, the ſide of which is not greater than half the given line.



Let AB be the given line, and C the ſide of the given ſquare; it is required to divide AB into two ſuch parts that their rectangle may be equal to the ſquare of C.

Upon AB deſcribe the ſemicircle BDA, and make BF perpendicular to AB (I. 11.), and equal to C. (I. 3.)

Through F draw FD parallel to AB (I. 27.); and from the point D where it cuts the circle, let fall the perpendicular DE (I. 12.); and AB will be divided at E as was required.

For ſince BDA is a ſemicircle (*by Conſt.*), and DE is perpendicular to the diameter AB (*by Conſt.*), the rectangle of AE, EB will be equal to the ſquare of ED (VI. 7. *Cor.*)

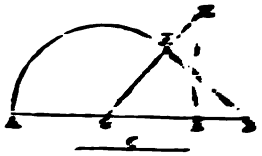
But ED is equal to FB (I. 30.) or C; whence the rectangle of AE, EB will be equal to the ſquare of C as was to be ſhewn.

SCHOLIUM. When BF, or C, is equal to half AB, FD will be a tangent to the circle, and the rectangle of AE, EB will be the greateſt poſſible.

PROP.

PROP. XXIII. PROBLEM.

To a given right line to add another right line such, that the rectangle of the whole and the part added shall be equal to a given square.



Let AB be the given line, and C the side of the given square; it is required to add a line to AB such, that the rectangle of the whole and the part added shall be equal to the square of C.

Make BE perpendicular to AB (I. 11.), and equal to C (I. 3.); also bisect AB in G (I. 10.), and join GE.

Then, if AB be produced, and GD be taken equal to GE (I. 3.), the part BD will be added to AB, as was required.

For on AB describe the semicircle BFA, cutting GE in F, and join FD.

Then, since the two sides GB, GE of the triangle GBE, are equal to the two sides GF, GD, of the triangle GFD, and the angle G is common, the angle GBE will be equal to the angle GFD, and the side FD to the side BE (I. 4.).

But the angle GBE is a right angle by Const. therefore the angle GFD is also a right angle; and consequently FD is a tangent to the circle AFB [illegible].

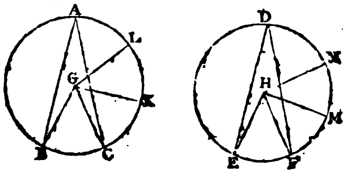
And ſince DF is a tangent to the circle, and DA is drawn to the oppoſite part of the circumference, the rectangle of AD, DB will be equal to the ſquare of DF (III. 29.)

But DF has been ſhewn to be equal to BE or C; whence the rectangle of AD, DB will alſo be equal to the ſquare of C.

Q. E. D.

PROP. XXIV. THEOREM.

Angles at the centres or circumferences of equal circles, have the ſame ratio with the arcs on which they ſtand.



Let ABC, DEF be two equal circles, in which BGC, EHF are angles at the centre, and BAC, EDF angles at the circumference; then will the arc BC be to the arc EF as the angle BGC is to the angle EHF, or as the angle BAC to the angle EDF.

For on the circumference of the circle ABC take any number of arcs whatever CK, KL each equal to BC; and on the circumference of the circle DEF any number of arcs whatever FM, MN, each equal to EF; and join GK, GL, HM, HN.

Then, becauſe the arcs BC, CK, KL are all equal to each other, the angles BGC, CGK, KGL will alſo be equal to each other (III. 21.)

And, therefore, whatever multiple the arc BL is of the arc BC, the ſame multiple will the angle BGL be of the angle BGC.

For the ſame reaſon, whatever multiple the arc EN is of the arc EF, the ſame multiple will the angle EHN be of the angle EHF.

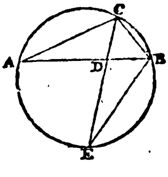
If, therefore, the arc BL be equal to the arc EN, the angle BGL will be equal to the angle EHN; and if equal, equal; and if leſs, leſs.

But BL and BGL are any equimultiples whatever of BC and BGC, and EN and EHN of EF and EHN; whence the arc BC is to the arc EF as the angle BGC is to the angle EHF (V. 5.)

And ſince the angle BGC is double the angle BAC, and the angle EHF is double the angle EDF (III. 14.), the arc BC will alſo be to the arc EF as the angle BAC is to the angle EDF (V. 13.) Q. E. D.

PROP. XXV. THEOREM.

The rectangle of the two ſides of any triangle, is equal to the rectangle of the ſegments of the baſe, made by a line biſecting the vertical angle, together with the ſquare of that line.



Let ABC be a triangle, having the angle ACB biſected by the right line CD; then will the rectangle of AC, CB be equal to the rectangle of AD, DB, together with the ſquare of CD.

For, about the triangle ABC, deſcribe the circle ABC (IV. 5.), cutting CD, produced, in E; and join EB.

Then, becauſe the angle ACD is equal to the angle ECB (*by Hyp.*), and the angle CAD to the angle CEB (III. 15.), the remaining angle ADC will be equal to the remaining angle CBE (I. 28. *Cor.*)

The triangles CAD, CEB being, therefore, equiangular, CA will be to CD as CE to CB (VI. 5.); and conſequently the rectangle of CA, CB is equal to the rectangle of CE, CD (VI. 12.)

But the rectangle of CE, CD is equal to the rectangle of ED, DC, together with the ſquare of CD (II. 10.); whence the rectangle of CA, CB is alſo equal to the rectangle of ED, DC, together with the ſquare of CD.

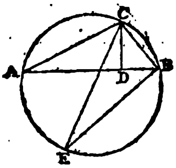
And ſince the rectangle of ED, DC is equal to the rectangle of AD, DB (III. 27.), the rectangle of AC, CB is alſo equal to the rectangle of AD, DB, together with the ſquare of CD.

Q. E. D.

PROP.

PROP. XXVI. THEOREM.

The rectangle of the two ſides of any triangle, is equal to the rectangle of the perpendicular, drawn from the vertical angle to the baſe, and the diameter of the circumſcribing circle.



Let ABC be a triangle, having CD perpendicular to AB; then will the rectangle of AC, CB be equal to the rectangle of CD and the diameter of the circumſcribing circle.

For, about the triangle ABC, deſcribe the circle AEC (IV. 5.); in which draw the diameter CE; and join EB.

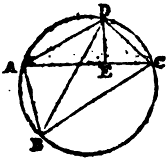
Then, ſince the angle CAD is equal to the angle CEB (III. 15.) and the angle ADC to the angle EBC, being each of them right angles (*Conſt. and* III. 16.), the remaining angle ACD will be equal to the remaining angle ECB (I. 28. *Cor.*)

The triangles ACD, ECB are, therefore, equiangular; whence AC is to CD as CE is to CB (VI. 5.); and conſequently the rectangle of AC, CB is equal to the rectangle of CD, CE (VI. 12.) Q. E. D.

SCHOLIUM. When ABC is an obtuſe angle, the perpendicular CD falls without the circle; but the ſame demonſtration will hold.

PROP. XXVII. THEOREM.

The rectangle of the two diagonals of any quadrilateral, inſcribed in a circle, is equal to the ſum of the rectangles of its oppoſite ſides.



Let ABCD be any quadrilateral inſcribed in a circle, of which the diagonals are AC, BD; then will the rectangle of AC, BD be equal to the rectangles of AB, DC and AD, BC.

For make the angle CDE equal to the angle ADB (I. 20.); then, if to each of theſe angles, there be added the common angle EDB, the angle ADE will be equal to the angle CDB.

The angle DAE is alſo equal to the angle DBC, being angles in the ſame ſegment, whence the remaining angle AED is equal to the remaining angle BCD (I. 28. *Cor.*)

Since, therefore, the triangles ADE, BDC are equiangular, AD is to AE as BD is to BC (VI. 5.); and conſequently the rectangle of AD, BC is equal to the rectangle of AE, BD (VI. 12.)

Again, the angle CDE being equal to the angle ADB (*by Conſt.*), and the angle ECD to the angle ABD (III. 15.), the remaining angle CED will be equal to the remaining angle BAD (I. 28. *Cor.*)

The triangles CED, ADB are, therefore, alſo equiangular; whence AB is to BD as EC is to DC (VI. 5.); and conſequently the rectangle of AB, DC is equal to the rectangle of EC, BD. (VI. 12.)

And if, to theſe equals, there be added the former, the rectangle of AB, DC together with the rectangle of AD, BC will be equal to the rectangle of EC, BD together with the rectangle of AE, BD.

But the rectangles of AE, BD, and EC, BD are equal to the rectangle of AC, BD (II. 8.); whence the rectangle of AC, BD is alſo equal to the rectangles of AB, DC and AD, BC.

Q. E. D.

BOOK VII.

DEFINITIONS.

1. The common ſection of two planes, is the line in which they meet, or cut each other.

2. A right line is perpendicular to a plane, when it is perpendicular to every right line which meets it in that plane.

3. A plane is perpendicular to a plane, when every right line in the one, which is perpendicular to their common ſection, is perpendicular to the other.

4. The inclination of a right line to a plane, is the angle it makes with another line, drawn from the point of ſection, to that point in the plane, which is cut by a perpendicular falling from any part of the former.

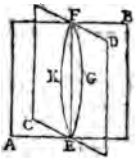
5. The inclination of a plane to a plane, is the angle contained by two right lines, drawn from any point in the common ſection, at right angles to that ſection; one in one plane, and the other in the other.

6. Parallel planes, are ſuch as being produced ever ſo far both ways will never meet.

7. A plane is ſaid to be extended by, or to paſs through a right line, when every part of that line lies in the plane.

PROP. I. Theorem.

The common ſection of any two planes is a right line.



Let AB, CD be two planes, whoſe common ſection is EF; then will EF be a right line.

For if not, let FGE be a right line, drawn in the plane AB; and FKE another right line, drawn in the plane CD.

Then, ſince the lines FGE, FKE are in different planes, they muſt fall wholly without each other.

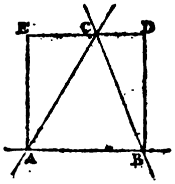
But the line FGE, having the ſame extremities with the line FKE, will coincide with it: whence they coincide and fall wholly without each other, at the ſame time, which is abſurd.

The lines FGE, FKE cannot, therefore, be right lines; and conſequently the line EF, which lies in each of the planes, muſt be a right line, as was to be ſhewn.

Scholium. One part of a right line cannot be in a plane, and another part out of it. For ſince the line can be produced in that plane, the part out of the plane, and the part produced would have different directions, which is abſurd.

PROP. II. THEOREM.

Any three right lines which mutually interſect each other, are all in the ſame plane.



Let AB, BC, CA be three right lines, which interſect each other in the points A, B, C; then will thoſe lines be in the ſame plane.

For let any plane AD paſs through the points A, B, and be turned round that line, as an axis, till it paſs through the point C.

Then, becauſe the points A, C are in the plane AD, the whole line AC muſt alſo be in it; or otherwiſe its parts would not lie in the ſame direction.

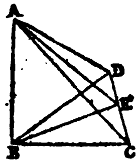
And, becauſe the points B, C are alſo in this plane, the whole line BC muſt likewiſe be in it; for the ſame reaſon.

But the line AB is in the plane AD, by hypotheſis; whence the three lines AB, BC, CA are all in the ſame plane, as was to be ſhewn.

COR. Any two right lines which interſect each other, are both in the ſame plane; and through any three points a plane may be extended.

PROP. III. THEOREM.

If a right line be perpendicular to two other right lines, at their point of interſection, it will alſo be perpendicular to the plane which paſſes through thoſe lines.



Let the right line AB be perpendicular to each of the two right lines BC, BD, at their point of interſection B; then will it alſo be perpendicular to the plane which paſſes through thoſe lines.

For make BD equal to BC; and, in the plane which paſſes through thoſe lines, draw any right line BE; and join the points CD, AD, AE and AC:

Then becauſe the ſide BC is equal to the ſide BD (*by Conſt.*), and the perpendicular AB is common to each of the triangles ABC, ABD, the ſide AD will alſo be equal to the ſide AC (I. 4.)

And ſince the triangles CAD, CBD are iſoſceles, the rectangle of CE, ED, together with the ſquare of EB, is equal to the ſquare of DB; and the rectangle of CE, ED together with the ſquare of EA, is equal to the ſquare of AD (II. 20.)

From each of theſe equals, take away the rectangle of CE, ED which is common, and the difference of the ſquares of EB, EA will be equal to the difference of the ſquares of DB, AD.

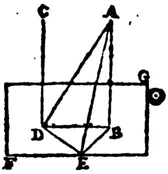
But the difference of the ſquares of DB, AD is equal to the ſquare of AB (II. 14 *Cor.*); whence the difference of the ſquares of EB, EA will alſo be equal to the ſquare of AB; and conſequently AB is perpendicular to BE, as was to be ſhewn.

COROLL. If a right line be perpendicular to three other right lines, at their point of interſection, thoſe lines will be all in the ſame plane.

For if either of them, as BE, were above or below the plane which paſſes through the other two, the angle ABE would be leſs or greater than a right angle.

PROP. IV. THEOREM.

If two right lines be perpendicular to the ſame plane, they will be parallel to each other.



Let the right lines AB, CD be each of them perpendicular to the plane FG, then will thoſe lines be parallel to each other.

For join the points D, B; and, in the plane FG, make BE perpendicular to DB, and equal to AB (I. 11. 3.); and join the points AE, AD.

Then, ſince the right lines AB, CD are perpendicular to the plane FG (*by Hyp.*), the angles ABD, ABE, CDB and CDE will be right angles (VII. *Def.* 2.)

And becauſe the ſide AB, is equal to the ſide ED (*by Conſt.*), the ſide DB common to each of the triangles BAD, BED, and the angles ABD, BDE right angles (*by Hyp. and Conſt.*), the ſide AD will alſo be equal to the ſide EB (I. 4.)

Again, ſince the ſides AD, DE are equal to the ſides EB, BA, and the ſide AE is common to each of the triangles EBA, EDA, the angle ADE will alſo be equal to the angle ABE (I. 7.), and is, therefore, a right angle.

And, becauſe the line ED is at right angles with each of the three lines DA, DB, DC, thoſe lines, together with the line AB, will be all in the ſame plane (VII. 3. *Cor.*)

Since, therefore, the lines AB, BD, DC are all in the ſame plane, and the angles ABD, CDB are each of them a right angle, the line AB will be parallel to the line CD (I. 23.), as was to be ſhewn.

COR. Any two parallel right lines AB, CD, are in the ſame plane; and any right line DA, which interſects thoſe parallels, is in the ſame plane with them.

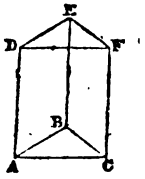
PROP.

And, in like manner, it may be proved that the line CD is alſo perpendicular to the plane HGK.

But, when two right lines are perpendicular to the ſame plane, they are parallel to each other (VII. 4.); whence the line AB is parallel to the line CD, as was to be ſhewn.

PROP. VII. THEOREM.

If two right lines that meet each other, be parallel to two other right lines that meet each other, though not in the ſame plane with them, the angles contained by thoſe lines will be equal.



Let the two right lines AB, BC, which meet each other in the point B, be parallel to the two right lines DE, EF which meet each other in the point E; then will the angle ABC be equal to the angle DEF.

For make BA, BC, ED, EF all equal to each other (I. 3.), and join AD, CF, BE, AC and DF.

Then, becauſe BA is equal and parallel to ED (*by Hyp.*), AD will be equal and parallel to BE (I. 29.)

And, for the ſame reaſon, CF will alſo be equal and parallel to BE.

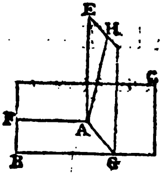
But lines which are equal and parallel to the ſame line, though not in the ſame plane with it, are equal and parallel to each other (I. 26. *and* VII. 6.); whence AD is equal and parallel to CF.

And ſince lines which join the correſponding extremes of two equal and parallel lines are alſo equal and parallel (I. 29.), AC will be equal and parallel to DF.

The three ſides of the triangle ABC are, therefore, equal to the three ſides of the triangle DEF, each to each; whence the angle ABC is equal to the angle DEF (I. 7.), as was to be ſhewn.

PROP. VIII. PROBLEM.

To draw a right line perpendicular to a given plane, from a given point in the plane.



Let A be the given point, and BC the given plane; it is required to draw a right line from the point A that ſhall be perpendicular to the plane BC.

Take any point E above the plane BC, and join EA; and through A draw AF, in the plane BC, at right angles with EA (I. 11.); then if EA be alſo at right angles with any other line which meets it in that plane; the thing required is done.

But if not, in the plane BC, draw AG at right angles to AF (I. 11.); and in the plane EG, which paſſes through the points E, A, G, make AH perpendicular to

AG (I. 11.); and it will alſo be perpendicular to the plane BC, as was required.

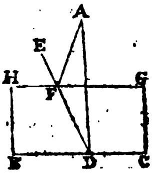
For, ſince the right line FA is perpendicular to each of the right lines AE, AG (*by Conſt.*), it will alſo be perpendicular to the plane EG which paſſes through thoſe lines (VII. 3.)

And becauſe a right line which is perpendicular to a plane is perpendicular to every right line which meets it in that plane (*Def.* 2.), FA will be perpendicular to AH.

But AG is alſo perpendicular to AH (*by Conſt.*); whence AH, being perpendicular to each of the right lines FA, AG, it will alſo be perpendicular to the plane BC (VII. 3.), as was to be ſhewn.

PROP. IX. PROBLEM.

To draw a right line perpendicular to a given plane, from a given point above it.



Let A be the given point, and BG the given plane; it is required to draw a right line from the point A that ſhall be perpendicular to the plane BG.

Take any right line BC, in the plane BG, and draw AD perpendicular to BC (I. 11.); then if it be alſo perpendicular to the plane BG, the thing required is done.

But if not, draw DE, in the plane BC, at right angles to BC (I. 11.), and make AF perpendicular to DE (I. 12.);

then will AF be perpendicular to the plane BG, as was required.

For, through the point F, draw the line HG parallel to the line BC (I. 27.)

Then ſince the right line BC is perpendicular to each of the right lines DA, DE, it will alſo be perpendicular to the plane which paſſes through thoſe lines (VII. 3.)

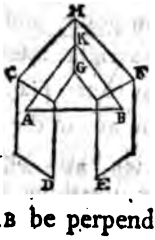
And becauſe the lines BC, HG are parallel to each other, and one of them, BC, is perpendicular to the plane ADF, the other, HG, will alſo be perpendicular to that plane (VII. 5.)

But if a line be perpendicular to a plane it will be perpendicular to all the lines which meet it in that plane (VII. *Def.* 2.); whence the line HG is perpendicular to AF.

And ſince the line AF is perpendicular to each of the lines HG, ED, at their point of interſection F, it will alſo be perpendicular to the plane BG (VII. 3.), as was to be ſhewn.

PROP. X. THEOREM.

Planes to which the ſame right line is perpendicular, are parallel to each other.



Let the right line AB be perpendicular to each of the planes CD, EF; then will thoſe planes be parallel to each other.

For if they be not, let them be produced till they meet each other; and in the line GH, which is their common ſection, take any point K; and join KA, KB:

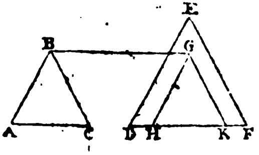
Then, becauſe the line AB is perpendicular to the plane EF (*by Hyp.*), it will alſo be perpendicular to the line BK, which lies in that plane (VII. *Def.* 2.); and the angle ABK will be a right angle.

And, for the ſame reaſon, the line AB, which is perpendicular to the plane DC (*by Hyp.*), will be perpendicular to the line AK; and the angle BAK will alſo be a right angle.

The angles ABK, BAK are, therefore, equal to two right angles, which is abſurd (I. 28.); and conſequently the planes can never meet, but muſt be parallel to each other (VII. *Def.* 6.), as was to be ſhewn.

PROP. XI. THEOREM.

If two right lines which meet each other, be parallel to two other right lines which meet each other, though not in the ſame plane with them, the planes which paſs through thoſe lines will be parallel.



Let the right lines AB, BC, which meet each other in B, be parallel to the right lines DE, EF, which meet each other in E, though not in the ſame plane with them; then will the plane ABC be parallel to the plane DEF.

For through the point B draw BG perpendicular to the plane DFE (VII. 9.); and make GH parallel to DE, and GK to EF (I. 27.)

Then becauſe BG is at right angles with the plane DFE, it will alſo be at right angles with each of the lines GH, GK which meet it in that plane (*Def.* 2.)

And ſince GH is parallel to DE or AB (*by Conſt. and* VII. 6.), and BG interſects them, the angles BGH, GBA are, together, equal to two right angles (I. 25.)

But the angle BGH has been ſhewn to be a right angle; whence the angle GBA is alſo a right angle; and conſequently GB is perpendicular to BA.

And, in the ſame manner, it may be ſhewn, that GB is perpendicular to BC.

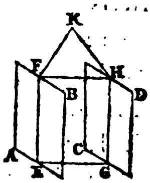
The right line GB, therefore, being perpendicular to each of the right lines BA, BC, will alſo be perpendicular to the plane ACB through which they paſs (VII. 3.)

But planes to which the ſame right line is perpendicular are parallel to each other (VII. 10.); whence the plane ACB is parallel to the plane DFE.

Q. E. D.

PROP. XII. THEOREM.

If any two parallel planes be cut by another plane, their common ſections will be parallel.



Let the two parallel planes AB, CD be cut by the plane EGHF; then will their common ſections EF, GH be parallel to each other.

For if EF, GH be not parallel, they may be produced till they meet, either on the ſide FH, or the ſide EG.

Let them be produced on the ſide FH, and meet each other in the point K.

Then, ſince the whole line EFK is in the plane AB, or the plane produced, the point K muſt be in that plane.

And becauſe the whole line GHK is in the plane CD, or the plane produced, the point K muſt alſo be in that plane.

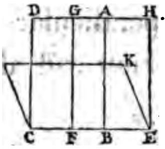
Since, therefore, the point K is in each of the planes AB, CD, thoſe planes, if produced, will meet in that point.

But the two planes are parallel to each other, by hypotheſis; whence they meet, and are parallel, at the ſame time, which is abſurd.

The lines EF, GH, therefore, do not meet on the ſide FH; and, in the ſame manner, it may be proved, that they do not meet on the ſide EG; conſequently they are parallel to each other. Q. E. D.

PROP. XIII. THEOREM.

If a right line be perpendicular to a plane, every plane which paſſes through it will alſo be perpendicular to that plane.



Let the right line AB be perpendicular to the plane CK; then will every plane which paſſes through that line be alſo perpendicular to CK.

For let ED be any plane which paſſes by the line AB; and in this plane draw any right line GF perpendicular to the common ſection CE (I. 11.)

Then, becauſe the line AB is perpendicular to the plane CK (*by Hyp.*), it will alſo be perpendicular to the line CE; and the angle ABF will be a right angle (VII. *Def.* 2.)

And ſince the angles ABF, GFB are each of them a right angle, and the lines AB, GF are in the ſame plane, they will be parallel to each other (VII. 4.)

Since, therefore, theſe lines are parallel to each other, and one of them, AB, is perpendicular to the plane CK,

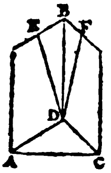
the

the other, GF, will alſo be perpendicular to that plane (VII. 5.)

But one plane is perpendicular to another, when any right line that can be drawn in it, at right angles to the common ſection, is alſo at right angles to the other plane (VII. *Def.* 3.); whence the plane BD is perpendicular to the plane CK, as was to be ſhewn.

PROP. XIV. THEOREM.

If two planes which cut each other, be each of them perpendicular to a third plane, their common ſection will alſo be perpendicular to that plane.



Let the two planes AB, CB be each of them perpendicular to the plane ACD; then will their common ſection BD be alſo perpendicular to ACD.

For if not, let DE be drawn in the plane AB, at right angles to the common ſection AD; and DF in the plane CB at right angles to the common ſection DC (I. 11.)

Then becauſe the plane AB is perpendicular to the plane ACD (*by Hyp.*), the line DE will alſo be perpendicular to that plane (VII. 3.)

And ſince the plane CB is perpendicular to the plane ACD (*by Hyp.*), the line DF will alſo be perpendicular to that plane (VII. 3.)

But lines which are perpendicular to the ſame plane are parallel to each other (VII. 4.); whence the lines DE, DF meet, and are parallel at the ſame time, which is abſurd.

Theſe lines, therefore, are not perpendicular to the plane ACD; and the ſame may be ſhewn of any other line but DB; whence DB is perpendicular to ACD, as was to be ſhewn.

BOOK VIII.

DEFINITIONS.

1. A ſolid angle is that which is made by three or more plane angles, which meet each other in the ſame point.

2. Similar ſolids, contained by plane figures, are ſuch as have all their ſolid angles equal, each to each, and are bounded by the ſame number of ſimilar planes.

3. A priſm is a ſolid whoſe ends are parallel, equal, and like plane figures, and its ſides parallelograms.

4. A parallelepiped is a priſm contained by ſix parallelograms, every oppoſite two of which are equal, alike, and parallel.

5. A rectangular parallelepiped is that whoſe bounding planes are all rectangles, which are perpendicular to each other.

6. A cube is a priſm, contained by ſix equal ſquare ſides, or faces.

7. A pyramid is a ſolid whoſe baſe is any right lined plane figure, and its ſide triangles, which meet each other in a point above the baſe, called the vertex.

8. A cylinder is a ſolid generated by the revolution of a right line about the circumferences of two equal and parallel circles, which remain fixed.

9. The axis of a cylinder is the right line joining the centres of the two parallel circles, about which the figure is deſcribed.

10. A cone is a ſolid generated by the revolution of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.

11. The axis of a cone is the right line joining the vertex, or fixed point, and the centre of the circle about which the figure is deſcribed.

12. Similar cones and cylinders are ſuch as have their altitudes and the diameters of their baſes proportional.

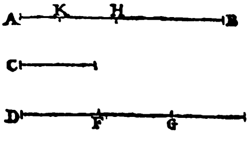
13. A ſphere is a ſolid generated by the revolution of a ſemi-circle about its diameter, which remains fixed.

14. The axis of a ſphere is the right line about which the ſemi-circle revolves; and the centre is the ſame as that of the ſemi-circle.

15. The diameter of a ſphere is any right line paſſing through the centre, and terminated both ways by the ſurface.

PROP. I. LEMMA.

If from the greater of two magnitudes, there be taken more than its half; and from the remainder, more than its half; and ſo on: there will at length remain a magnitude leſs than the leaſt of the propoſed magnitudes.



Let AB and C be any two magnitudes, of which AB is the greater; then, if from AB there be taken more than its half; and from the remainder more than its half; and ſo on: there will at length remain a magnitude leſs than C.

For ſince AB and C are each finite magnitudes, it is evident that C may be taken ſuch a number of times as at length to become greater than AB.

Let, therefore, DE be ſuch a multiple of C as is greater than AB, and divide it into the parts DF, FG, GE, each equal to C.

Alſo from AB take BH greater than its half; and from the remainder AH, take HK greater than its half, and ſo on, till there be as many diviſions in AB as there are in DE.

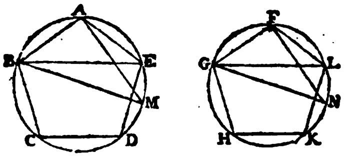
Then becauſe DE is greater than AB; and BH, taken from AB, is greater than its half, but EG, taken from DE, is not greater than its half; the remainder GD will be greater than the remainder HA.

And, again, becauſe GD is greater than HA, and HK, taken from HA, is greater than its half, but GF, taken from GD, is not greater than its half; the remainder FD will be greater than the remainder AK.

But FD is equal to C by conſtruction, whence C is greater than AK; or, which is the ſame thing, AK is leſs than C, as was to be ſhewn.

PROP. II. THEOREMS.

Similar polygons inſcribed in circles, are to each other as the ſquares of the diameters of thoſe circles.



Let ABCDE, FGHKL be two ſimilar polygons, inſcribed in the circles ABD, FGK: then will the polygon ABCDE be to the polygon FGHKL as the ſquare of the diameter BM is to the ſquare of the diameter GN.

For join the points B, E and A, M, G, L and F, N:

Then, becauſe the polygon ABCDE is ſimilar to the polygon FGHKL (*by Hyp.*), the angle BAE is equal to the angle GFL, and BA is to AE, as GF is to FL. (VI. *Def.* 1.)

And, ſince the angle BAE, of the triangle ABE, is equal to the angle GFL, of the triangle FGL, and the ſides about thoſe angles are proportional, the angle AEB will alſo be equal to the angle FLG (VI. 5.)

But the angle AEB is equal to the angle AMB, and the angle FLG to the angle FNG (III. 15.), conſequently the angle AMB is alſo equal to the angle FNG.

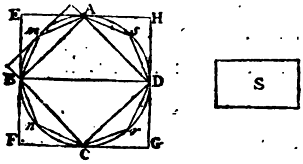
And ſince theſe angles are equal to each other, and the angles BAM, GFN are each of them right angles (III. 16.), the angle MBA will alſo be equal to the angle NGF (I. 28. *Cor.*), and BM will be to GN as BA is to GF (VI. 5.)

But the polygon ABCDE is to the polygon FGHKL as the ſquare of BA is to the ſquare of GF (VI. 17.), therefore the polygon ABCDE is alſo to the polygon FGHKL as the ſquare of BM is to the ſquare of GN.

Q. E. D.

PROP. III. THEOREM.

A polygon may be inſcribed in a circle that ſhall differ from it by leſs than any aſſigned magnitude whatever.



Let ABCD be a circle, and S any given magnitude whatever; then may a polygon be inſcribed in the circle ABCD that ſhall differ from it by leſs than the magnitude S.

For, let AC, EG be two ſquares, the one deſcribed in the circle ABCD, and the other about it (IV. 6, 7.); and biſect the arcs AB, BC, CD, DA, in the points m, n, r and s (III. 33.); and join Am, mB, Bn, nC, Cr, rD, Ds and sA:

Then ſince the ſquare AC is half the ſquare EG (I. 32.), and the ſquare EG is greater than the circle ABCD, the ſquare AC will be greater than half the circle ABCD.

In like manner, if tangents be drawn to the circle through the points m, n, r, s, and parallelograms be deſcribed upon the right lines AB, BC, CD, DA, the triangles A*m*B, B*n*C, C*r*D, D*s*A will each of them be half the parallelogram in which it ſtands (I. 32.)

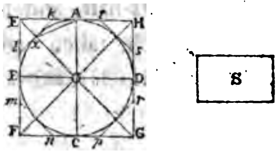
But every ſegment is leſs than the parallelogram which circumſcribes it; and therefore each of the triangles A*m*B, B*n*C, C*r*D, D*s*A is greater than half the ſegment of the circle which contains it.

And, if each of the arcs A*m*, *m*B, &c. be again divided into two equal parts, and right lines be drawn to the points of biſection, the triangles thus formed, may in like manner, be ſhewn to be greater than half the ſegments which contain them; and ſo on continually.

Since, therefore, the circle ABCD is greater than the ſpace S, and from the former there has been taken more than its half, and from the remainder more than its half, &c. there will at length remain ſegments which, taken together, ſhall be leſs than the exceſs of the circle ABCD above the ſpace S (VIII. 1.), as was to be ſhewn.

PROP. IV. THEOREM.

A polygon may be circumſcribed about a circle that ſhall differ from it by leſs than any aſſigned magnitude whatever.



Let ABCD be the circle, and S any given magnitude whatever; then may a polygon be circumſcribed about the circle ABCD, that ſhall differ from it by leſs than the magnitude S.

For let the circle ABCD be circumſcribed by the ſquare EFGH (IV. 7.), and biſect the arcs AB, BC, CD, DA with the lines OE, OF, OG and OH; and to the points of biſection draw the tangents *kl*, *mn*, *pr*, *st* (III. 10.)

Then ſince *kl* is a tangent to the circle, and OE is drawn from the centre through the point of contact, the angle E*xk* is a right angle (III. 12.), and E*k* will be greater than *kx* (I. 17.) or its equal *k*A.

But triangles of the ſame altitude are to each other as their baſes (VI. 1.); whence the baſe E*k* being greater than the baſe *k*A, the triangle E*xk* will alſo be greater than the triangle *kx*A.

And becauſe the triangle E*xk* is greater than the triangle *kx*A, it will alſo be greater than half the curvelineal ſpace E*x*A: and the ſame may be ſhewn of any other triangle and the curvelineal ſpace to which it belongs.

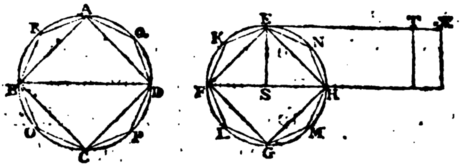
In like manner, if the arcs A*x*, *x*B, &c. be again biſected, and tangents be drawn to the points of biſection, the triangles thus formed will be greater than half the curvelineal ſpaces to which they belong.

Since, therefore, the exceſs of the ſquare above the circle is greater than the magnitude S, and from the former there has been taken more than its half, and from the remainder more than its half, and ſo on, there will at length remain ſpaces, which, taken together, ſhall be leſs than the magnitude S (VIII. 1.), as was to be ſhewn.

PROP.

PROP. V. THEOREMS.

Circles are to each other as the ſquares of their diameters.



Let ABCD, EFGH be two circles, and BD, FH their diameters: then will the ſquare of BD be to the ſquare of FH as the circle ABCD is to the circle EFGH.

For, if they have not this ratio, the ſquare of BD will be to the ſquare of FH, as the circle ABCD is to ſome ſpace either leſs or greater than the circle EFGH.

Firſt, let it be to a ſpace ST leſs than the circle EFGH; and inſcribe the two ſimilar polygons AROPQ, EKLMN ſo that the circle EFGH may exceed the latter by leſs than it exceeds the ſpace ST (VIII. 3.)

Then, ſince the circle EFGH exceeds the polygon EKLMN by leſs than it exceeds the ſpace ST, the polygon EKLMN will be greater than the ſpace ST.

And, becauſe ſimilar polygons, inſcribed in circles, are to each other as the ſquares of their diameters (VIII. 2.), the ſquare of BD will be to the ſquare of FH as the polygon AROPQ is to the polygon EKLMN.

But the ſquare of BD is alſo to the ſquare of FH as the circle ABCD is to the ſpace ST (*by Conſt.*); whence the circle ABCD will be to the ſpace ST, as the polygon AROPQ is to the polygon EKLMN.

The circle ABCD, therefore, being greater than the polygon AROPQ, which is contained in it, the ſpace ST will alſo be greater than the polygon EKLMN.

It is, therefore, leſs and greater at the ſame time, which is impoſſible; conſequently the ſquare of BD is not to the ſquare of FH as the circle ABCD is to any ſpace leſs than the circle EFGH.

And, in the ſame manner, it may be demonſtrated, that the ſquare of FH is ~~not~~ the ſquare of BD as the circle EFGH is to any ſpace leſs than the circle ABCD.

Nor, is the ſquare of BD to the ſquare of FH as the circle ABCD is to a ſpace greater than the circle EFGH.

For, if it be poſſible, let it be ſo to the ſpace SX, which is greater than the circle EFGH.

Then, ſince the ſquare of BD is to the ſquare of FH as the circle ABCD is to the ſpace SX, therefore, alſo, inverſely, the ſquare of FH is to the ſquare of BD as the ſpace SX is to the circle ABCD (V. 7.)

But the ſpace SX is greater than the circle EFGH (*by Hyp.*); whence the ſpace SX is to the circle ABCD as the circle EFGH is to ſome ſpace leſs than the circle ABCD (V. 14.)

The ſquare of FH is, therefore, to the ſquare of BD as the circle EFGH is to a ſpace leſs than the circle ABCD (V. 11.), which has been ſhewn to be impoſſible.

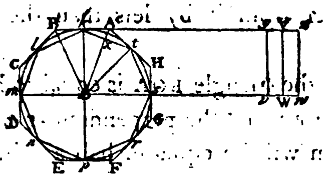
Since, therefore, the ſquare of BD is not to the ſquare of FH as the circle ABCD is to any ſpace either leſs or greater than the circle EFGH, the ſquare of BD muſt be to the ſquare of FH as the circle ABCD is to the circle EFGH. Q.E.D.

COR. 1. Circles are to each other as the ſquares of their radii; theſe being half the diameters.

Cor. 2. If the radii or diameters of three circles be reſpectively equal to the three ſides of a right angled triangle, that whoſe radius or diameter is the hypothenuſe will be equal to the other two taken together (II. 14.)

PROP. VI. Theorem.

Every circle is equal to the rectangle of its radius, and a right line equal to half its circumference.



Let *k m p s* be a circle, and OV a rectangle contained under the radius O*k* and a right line OW equal to half the circumference; then will the circle *k m p s* be equal to the rectangle OV.

For if it be not, it muſt be either greater or leſs.

Let it be greater; and let the rectangle O*z* be equal to the circle *k m p s*; and inſcribe a polygon *l n r t* in the circle *k m p s* that ſhall differ from it by leſs than the magnitude W*z* (VIII. 3.)

Then ſince the triangle *k o t* is equal to half a rectangle under the baſe *k t* and the perpendicular O*x* (I. 32.), the whole polygon will be equal to half a rectangle under its perimeter and the perpendicular O*x*.

And becauſe OW is greater than half the perimeter of any polygon that can be inſcribed in the circle *k m p s* (*by Hyp.*), and o*k* is greater than o*x* (I. 17.), the rectangle ov will alſo be greater than the polygon *l n r t*.

But the polygon differs from the circle, or from the rectangle o*z*, by leſs than the magnitude w*z* (*by Conſt.*), and ov differs from oz by wz; conſequently the polygon is greater than the rectangle ov.

It is, therefore, both greater and leſs at the ſame time, which is abſurd; whence the circle *k m p s* is not greater than the rectangle ov.

Again, let it be leſs than ov, by the rectangle w*y*; and let BDFH be a polygon circumſcribed about the circle, that ſhall differ from it by leſs than the magnitude w*y* (VIII. 4.)

Then ſince the triangle BOA is equal to half a rectangle under the baſe BA and the perpendicular o*k* (I. 32.), the whole polygon will be equal to half a rectangle under its perimeter and perpendicular o*k*.

And becauſe ow is leſs than half the perimeter of any polygon that can be circumſcribed about the circle (*by Hyp.*), and o*k* is common, the rectangle ov will alſo be leſs than the polygon BDFH.

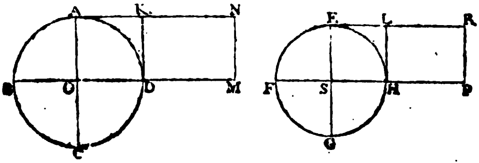
But the polygon differs from the circle, or from o*y*, by leſs than the magnitude w*y* (*by Hyp.*), and ov differs from o*y* by w*y*; conſequently the rectangle ov will be greater than the polygon BDFH, which is abſurd.

Since, therefore, the rectangle ov is neither greater nor leſs than the circle *k m p s*, it muſt be equal to it, as was to be ſhewn.

PROP.

PROP. VII. THEOREM.

The circumferences of circles are in proportion to each other as their diameters.



Let ABCD, EFGH be any two circles, whoſe diameters are BD, FH; then will the circumference ABCD be to the circumference EFGH as the diameter BD is to the diameter FH.

For let OM, SP be two right lines equal to the ſemicircumferences DAB, HEF, and on the radii OA, SE make the ſquares OK SL (II. 1.), and complete the rectangles ON, SR:

Then ſince the rectangles ON, SR are equal to the circles ABCD, EFGH (VIII. 6.), and the circles are to each other as the ſquares of their radii (VIII. 5. *Cor.*) the rectangle ON will alſo be to the ſquare OK as the rectangle SR is to the ſquare SL (V. 9.)

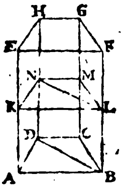
But ON is to OK as OM to OD (VI. 1.), and SR to SL as SP to SH (VI. 1.); therefore, by equality, OM will be to OD as SP is to SH (V. 11.)

And becauſe any equimultiples of four proportional quantities, are alſo proportional (V. 13.), twice OM will be to twice OD as twice SP is to twice SH; or, by alternation, twice OM is to twice SP as twice OD is to twice SH.

But twice OM and twice SP are equal to the circumferences ABCD, EFGH (*by Conſt.*); and twice OD and twice SH are equal to the diameters BD, FH; whence the circumference ABCD is to the circumference EFGH as the diameter BD is to the diameter FH. Q. E. D.

PROP. VIII. THEOREM.

If a priſm be cut by a plane parallel to its baſe, the ſection will be equal and like the baſe.



Let AG be a priſm, and KLMN a plane parallel to the baſe ABCD; then will KLMN be equal and like ABCD.

For join the points NL, and DB:

Then ſince KM, AC are parallel planes (*by Hyp.*), and the plane AN cuts them, the ſection KN will be parallel to the ſection AD (VII. 12.)

And ſince AK is alſo parallel to DN (VIII. *Def.* 3.), the figure AN is a parallelogram; and conſequently KN is equal to AD (I. 30.)

In like manner it may alſo be ſhewn, that KL is equal to AB, LM to BC, and MN to CD.

And ſince KN, KL in the plane KM, are parallel to AD, AB in the plane AC, the angle NKL will be equal to the angle DAB (VII. 7.)

The

The two ſides KN, KL of the triangle KLN, being, therefore, equal to the two ſides AD, AB of the triangle ABD, and the angle NKL to the angle DAB, the triangle KLN will be equal and like the triangle ABD (I. 4.)

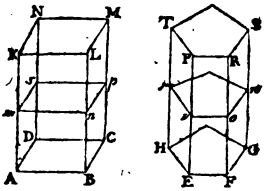
And in the ſame manner it may be ſhewn, that the triangle LMN is equal and like to the triangle BCD.

But the triangles KLN, LMN are, together, equal to the ſection KLMN; and the triangles ABD, BCD to the ſection ABCD; whence the ſection KLMN is equal and like to the ſection ABCD.

Q. E. D.

PROP. IX. THEOREM.

Priſms of equal baſes and altitudes are equal to each other.



Let AM, ES be any two priſms, ſtanding upon the equal baſes ABCD, EFGH, and having equal altitudes; then will AM be equal to ES.

For parallel to the baſes, and at equal diſtances from them, draw the planes *mp* and *vw*.

Then, by the laſt propoſition, the ſection *mnps* will be equal to the baſe ABCD, and the ſection *vowr* to the baſe EFGH.

But the baſe ABCD is equal to the baſe EFGH by hypotheſis; whence the ſection *mnps* is, alſo, equal to the ſection *vowr*.

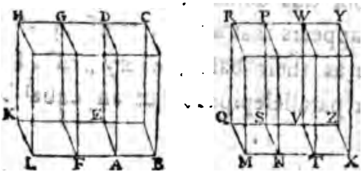
And in the ſame manner it may be ſhewn, that any other ſections, at equal diſtances from the baſes, are equal to each other.

Since therefore every ſection in the priſm AM is equal to its correſponding ſection in the priſm ES, the priſms themſelves, which are compoſed of thoſe ſections, muſt alſo be equal. Q. E. D.

COR. Every priſm is equal to a rectangular parallelepipedon of an equal baſe and altitude.

PROP. X. THEOREM.

Rectangular parallelepipedons, of equal altitudes, are to each other as their baſes.



Let AC, MP be two rectangular parallelepipedons, having the equal altitudes ED, QR; then will AC be to MP as the baſe BE is to the baſe NQ.

For in AB, produced, take any number of right lines AF, FL each equal to AB; and in MN, produced, take any number of right lines NT, TX each equal to MN.

Complete the parallelograms FE, FK, MV, TZ, and make the upright ſolids AG, FH, NW, TY of equal altitudes with AC or MP.

Then, becauſe AF, FL are each equal to AB, and NT, TX are each equal to MN (*by Conſt.*), the parallelograms FE, FK will be each equal to BE, and the parallelograms NV, TZ to NQ (II. 5.)

And, ſince the ſolids AG, FH have equal baſes and altitudes with the ſolid AC, they will be each equal to AC (VIII. 9.); and, for the ſame reaſon, the ſolids NW, TY, will be each equal to NR.

Whatever multiple, therefore, the baſe BK is of the baſe BE, the ſame multiple will the ſolid BH be of the ſolid AC; and, for the ſame reaſon, whatever multiple the baſe MZ is of the baſe NQ, the ſame multiple will the ſolid MY be of the ſolid NR.

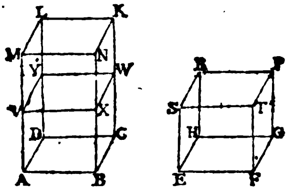
If, therefore, the baſe BK be equal to the baſe MZ, the ſolid BH will be equal to the ſolid MY; and if greater, greater; and if leſs, leſs; whence the baſe BE is to the baſe NQ, as the ſolid AC is to the ſolid NR (V. *Def.* 5.)

Q. E. D.

COR. From this demonſtration, and the Cor. to the laſt Prop. it appears that all priſms of equal altitudes, are to each other as their baſes; every priſm being equal to a rectangular parallelepipedon of an equal baſe and altitude.

PROP. XI. THEOREM.

Rectangular parallelepipedons of equal baſes are to each other as their altitudes.



Let AK, EP be two rectangular parallelepipedons ſtanding on the equal baſes AC, EG; then will AK be to EP as the altitude AM is to the altitude ES.

For let AW be a rectangular parallelepipedon on the baſe AC, whoſe altitude AV is equal to the altitude ES of the parallelepipedon EP:

Then, ſince the baſe AC is equal to the baſe EG (*by Hyp.*), and the altitude AV is equal to the altitude ES (*by Conſt.*), the ſolid AW will be equal to the ſolid EP (VIII. 9.)

And if AL, AY be conſidered as baſes, the ſolid AK will be to the ſolid AW as the baſe AL is to the baſe AY (VIII. 10.)

But the baſe AL is to the baſe AY as the ſide AM is to the ſide AV (VI. 1.); whence by equality the ſolid AK will be to the ſolid AW as the altitude AM is to the altitude AV (V. 11.)

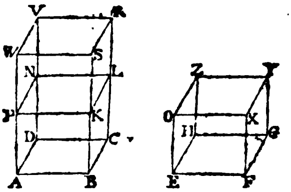
Since, therefore, the ſolid AW is equal to the ſolid EP, and the altitude AV to the altitude ES, the ſolid AK will alſo be to the ſolid EP as AM is to ES (V. 9.)

Q. E. D.

COR. From the reaſon given in the *Cor.* to the laſt Prop. it follows, that all priſms of equal baſes, are to each other as their altitudes.

PROP. XII. THEOREM.

The baſes and altitudes of equal rectangular parallelepipedons are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the parallelepipedons will be equal.



Let the rectangular parallelepipedon AR be equal to the rectangular parallelepipedon EY; then will the baſe AC be to the baſe EG, as the altitude EO is to the altitude AW.

For let AL be a rectangular parallelepipedon on the baſe AC, whoſe altitude AP is equal to EO, the altitude of the parallelepipedon EY.

Then ſince the altitudes AP, EO are equal to each other (*by Conſt.*), the ſolid AL will be to the ſolid EY as the baſe AC is to the baſe EG (VIII. 10.)

And becauſe the ſolid AR is equal to the ſolid EY (*by Hyp.*), the ſolid AL will be to the ſolid AR as AC is to EG (V. 9.)

But

But the ſolid AL is to the ſolid AR as AP is to AW (VIII. 11.); whence, alſo, AC is to EG as AP is to AW (V. 11.), or AC to EG as EO to AW.

Again, let AC be to EG as EO is to AW; then will AB be equal to EY.

For, ſince AL is to EY as AC to EG (VIII. 10.), and AC to EG as EO to AW (*by Hyp.*), AL will be to EY as EO to AW (V. 11.)

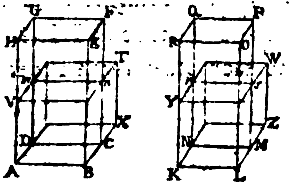
But EO, or AP, is to AW as AB is to AR (VIII. 11.); therefore AL will be to EY as AL is to AR (V. 11.)

And ſince the antecedents are equal, the conſequents will alſo be equal; whence the ſolid AR is equal to the ſolid EY, as was to be ſhewn.

COR. The ſame proportion will hold of priſms in general; theſe being equal to rectangular parallelepipedons of equal baſes and altitudes.

PROP. XIII. THEOREM.

Similar rectangular parallelepipedons are to each other as the cubes of their like ſides.



Let AF, KP be two ſimilar rectangular parallelepipedons, whoſe like ſides are AB, KL; then will AF be to KP as the cube of AB is to the cube of KL.

For let AT, KW be two cubes ſtanding on AX, KZ, the ſquares of the ſides AB, KL.

Then ſince parallelepipedons on the ſame baſe are to each other as their altitudes (VIII. 11.), AF will be to A*n* as AH to AV, or AB; and KP to K*s* as KR to KY, or KL.

But the planes ABEH, KLOR being ſimilar (VIII. *Def.* 2.), AH will be to AB as KR is to KL (VI. *Def.* 1.); whence AF is to A*n* as KP to K*s* (V. 11.); or AF to KP as A*n* to K*s* (V. 15.)

Again, ſince parallelepipedons of the ſame altitude are to each other as their baſes (VIII. 10.), AT will be to A*n* as AX to AC; and KW to K*s* as KZ to KM.

And becauſe AX, or the ſquare of AB, is to AC, as KZ, or the ſquare of KL, is to KM (VI. 17.); AT will be to A*n* as KW is to K*s* (V. 11.); or AT to KW as A*n* to K*s* (V. 15.)

But AF has been ſhewn to be to KP as A*n* is to K*s*; therefore, alſo, AF is to KP as AT to KW (V. 11.)

Q. E. D.

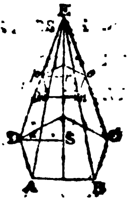
COR. 1. Similar rectangular parallelepipedons are to each other as the cubes of their altitudes; theſe being conſidered as like ſides of the ſolids.

COR. 2. Every priſm being equal to a parallelepipedon of an equal baſe and altitude (VIII. 9. *Cor.*), all ſimilar priſms will be to each other as the cubes of their altitudes, or like ſides.

PROP.

PROP. XIV. THEOREM.

If a pyramid be cut by a plane parallel to its baſe, the ſection will be to the baſe as the ſquares of their diſtances from the vertex.



Let EDABC be a pyramid, and *mo* a ſection parallel to the baſe AC; then will *mo* be to AC as the ſquares of their diſtances from the vertex.

For draw ES perpendicular to the plane of the baſe AC (VII. 9.); and join DS and *pr*.

Then, ſince *mp*, *mn* are parallel to AD, AB (VII. 12.), the angle *pnm* will be equal to the angle DAB (VII. 7.); and *pm* will be to DA as E*m* to EA, or as *mn* to AB (VI. 3.)

For a like reaſon each of the angles in the ſection *mo* are equal to their correſponding angles in the baſe AC, and the ſides about them are proportional; whence *mo* is ſimilar to AC (VI. *Def.* 1.)

And becauſe *pm* is parallel to DA, and *pr* to DS (VII. 12.), *pm* will be to DA as E*p* to ED, or as E*r* to ES (VI. 3.)

The lines *pm*, DA, E*r* and ES being, therefore, proportional, the ſquare of *pm* will be to the ſquare of DA, as the ſquare of E*r* is to the ſquare of ES (VI. 19. *Cor.*)

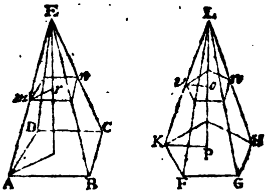
But the ſquare of *pm* is to the ſquare of DA as *mo* is to AC (VI. 17.); whence the ſquare of E*r* is to the ſquare of ES as *mo* is to AC (V. 11.)

Q. E. D.

COR. If a pyramid be cut by a plane parallel to its baſe, the ſection will be ſimilar to the baſe.

PROP. XV. THEOREM.

Pyramids of equal baſes and altitudes are equal to each other.



Let EDABC, LKFGH be any two pyramids, of which the baſe AC is equal to the baſe FH, and the altitude ES to the altitude LP; then will EDABC be equal to LKFGH.

For make E*r* equal to L*o*; and draw the ſections *mn*, *vw*, parallel to the baſes AC, FH.

Then, by the laſt propoſition, the ſquare of E*r* is to the ſquare of E*s* as *mn* is to AC; and the ſquare of L*o* to the ſquare of LP as *vw* is to FH.

And ſince the ſquare of E*r* is equal to the ſquare of L*o* (*Conſt.* and II. 2.); and the ſquare of ES to the ſquare of LP (*Hyp. and* II. 2.) *mn* will be to AC as *vw* is to FH (V. 9.)

But AC is equal to FH, by hypotheſis; whence *mn* is, alſo, equal to *vw* (V. 10.)

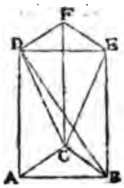
And, in the ſame manner, it may be ſhewn, that any other ſections, at equal diſtances from the vertices, are equal to each other.

Since, therefore, every ſection in the pyramid EDABC is equal to its correſponding ſection in the pyramid LKFGH, the pyramids themſelves, which are compoſed of thoſe ſections, muſt alſo be equal.

Q. E. D.

PROP. XVI. THEOREM.

Every pyramid of a triangular baſe, is the third part of a priſm of the ſame baſe and altitude.



Let DABC be a pyramid, and FDABE a priſm, ſtanding upon the ſame baſe ABC, and having the ſame altitude; then will DABC be a third of FDABE.

For in the planes of the three ſides of the priſm, draw the diagonals DB, DC and CE:

Then becauſe DB divides the parallelogram AE into two equal parts, the pyramid whoſe baſe is ABD, and vertex C, is equal to the pyramid whoſe baſe is BED and vertex C (VIII. 15.)

And ſince the oppoſite ends of the priſm are equal to each other (VIII. *Def.* 8.), the pyramid whoſe baſe is ABC and vertex D, is equal to the pyramid whoſe baſe is DEF and vertex C (VIII. 15.)

But the pyramid whoſe baſe is ABC and vertex D, is equal to the pyramid whoſe baſe is ABD and vertex C, being both contained by the ſame planes.

The three pyramids DABC, CBED and CEFD are, therefore, all equal to each other; and conſequently the priſm FDABE, which is compoſed of them, is triple the pyramid DABC, as was to be ſhewn.

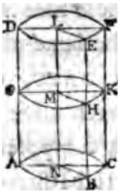
Cor. Every pyramid is the third part of a priſm of the ſame baſe and altitude; ſince the baſe of the priſm, whatever be its figure, may be divided into triangles, and the whole ſolid into triangular priſms, and pyramids.

Scholium. Whatever has been demonſtrated of the proportionality of priſms, holds equally true of pyramids; the former being always triple the latter.

PROP.

PROP. XVII. THEOREM.

If a cylinder be cut by a plane parallel to its baſe, the ſection will be a circle, equal to the baſe.



Let AF be a cylinder, and GHK a ſection parallel *to its* baſe ABC; then will GHK be a circle, equal to ABC.

For let the planes NE, NF paſs through the axis of the cylinder LN, and meet the ſection GHK in M, H and K.

Then, ſince the circle DEF is equal and parallel to the circle ABC (VIII. *Def.* 8.), the radii LF, LE will be equal and parallel to the radii NC, NB (III. 5. *and* VII. 12.)

And becauſe lines which join the correſponding extremes of equal and parallel lines are themſelves parallel (I. 29.), FC, EB will be parallel to LN; or KC, HB to MN.

In like manner, ſince the circle GHK is parallel to the circle ABC (*by Hyp.*), MK, MH will be parallel to NC, NB.

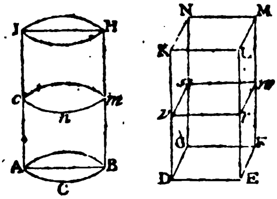
And, becauſe the oppoſite ſides of parallelograms are equal (I. 30.), MK will be equal to NC, and MH to NB.

But NC, NB are equal to each other, being radii of the ſame circle; whence MK, MH are alſo equal to each other.

And the ſame may be ſhewn of any other lines, drawn from the point M, to the circumference of the ſection GHK; conſequently GHK is a circle, and equal to ABC, as was to be ſhewn.

PROP. XVIII. THEOREM.

Every cylinder is equal to a priſm of an equal baſe and altitude.



Let AH be a cylinder, and DM a priſm, ſtanding upon equal baſes ACB, DEF, and having equal altitudes; then will AH be equal to DM.

For parallel to the baſes, and at equal diſtances from them, draw the planes *onm*, and *vrw*.

Then by the laſt Prop. and Prop. 8, the ſection *onm* is equal to the baſe ACB, and the ſection *vrw* to the baſe DEF.

But the baſe ACB is equal to the baſe DEF, by hypotheſis; whence the ſection *onm* is alſo equal to the ſection *vrw*.

And, in the ſame manner, it may be ſhewn, that any other ſections, at equal diſtances from the baſe, are equal to each other.

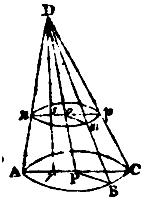
Since, therefore, every ſection of the cylinder is equal to its correſpondent ſection in the priſm, the ſolids themſelves, which are compoſed of thoſe ſections, muſt alſo be equal.

Q. E. D.

SCHOLIUM. Whatever has been demonſtrated of the proportionality of priſms, holds equally true of cylinders; the former being equal to the latter.

PROP. XIX. THEOREM.

If a cone be cut by a plane parallel to its baſe, the ſection will be to the baſe as the ſquares of their diſtances from the vertex.



Let DABC be a cone, and *nmp* a ſection parallel to the baſe ABC; then will *nmp* be to ABC as the ſquares of their diſtances from the vertex.

For draw the perpendicular D*r*; and let the planes CDP, BDP paſs through the axis of the cone, and meet the ſection in *o*, *p*, and *m*.

Then ſince the ſection *nmp* is parallel to the baſe ABC (*by Hyp.*), and the planes BO, CO cut them, *op* will be parallel to PC, and *om* to PB (VII. 12.)

And becauſe the triangles formed by theſe lines are equiangular, *om* will be to PB as D*o* to DP, or as *op* to PC (VI. 5.)

But PB is equal to PC, being radii of the ſame circle; wherefore *om* will alſo be equal to *op* (V. 10.)

And the ſame may be ſhewn of any other lines drawn from the point *o* to the circumference of the ſection *nmp*; whence *nmp* is a circle.

Again, by ſimilar triangles, D*s* is to D*r* as D*o* to DP, or as *om* to PB: whence the ſquare of D*s* is to the ſquare of D*r* as the ſquare of *om* is to the ſquare of PB (VI. 19.)

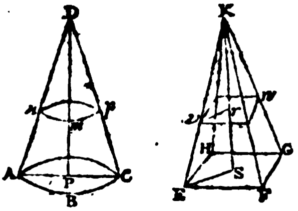
But the ſquare of *om* is to the ſquare of PB as the circle *nmp* is to the circle ABC (VIII. 5.); therefore the ſquare of D*s* is to the ſquare of D*r* as the circle *nmp* is to the circle ABC (V. 11.)

Q. E. D.

COR. If a cone be cut by a plane parallel to its baſe the ſection will be a circle.

PROP. XX. THEOREM.

Every cone is equal to a pyramid of an equal baſe and altitude.



Let DABC be a cone, and KEFGH a pyramid, ſtanding upon equal baſes ABC, EFGH, and having equal altitudes DP, KS; then will DABC be equal to KEFGH.

For parallel to the baſes, and at equal diſtances D*o*, K*r* from the vertices, draw the planes *nmp* and *vw*.

Then by the laſt Prop. and Prop. 18, the ſquare of D*o* is to the ſquare of DP as *nmp* is to ABC; and the ſquare of K*r* to the ſquare of KS as *vw* to EG.

And ſince the ſquares of D*o*, DP are equal to the ſquares of K*r*, KS. (*Conſt. and* II. 2.), *nmp* is to ABC as *vw* is to EG (V. 11.)

But ABC is equal to EG, by hypotheſis; wherefore *nmp* is, alſo, equal to *vw* (V. 10.)

And, in the ſame manner, it may be ſhewn, that any other ſections, at equal diſtances from the vertices, are equal to each other.

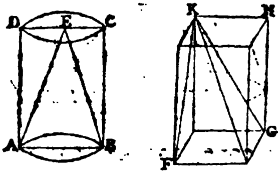
Since, therefore, every ſection in the cone is equal to its correſponding ſection in the pyramid, the ſolids DABC, KEFGH of which they are compoſed, muſt be equal.

Q. E. D.

PROP.

PROP. XXI. THEOREM.

Every cone is the third part of a cylinder of the ſame baſe and altitude.



Let EAB be a cone, and DABC a cylinder, of the ſame baſe and altitude; then will EAB be a third of DABC.

For let KFG, KFGH be a pyramid and priſm, having an equal baſe and altitude with the cone and cylinder.

Then ſince cylinders and priſms of equal baſes and altitudes are equal to each other (VIII. 18.) the cylinder DABC will be equal to the priſm KFGH.

And, becauſe cones and pyramids of equal baſes and altitudes are equal to each other (VIII. 20.), the cone EAB will be equal to the pyramid KFG.

But the pyramid KFG is a third part of the priſm KFGH (VIII. 16.), wherefore the cone EAB is, alſo, a third part of the cylinder DABC.

Q.E.D.

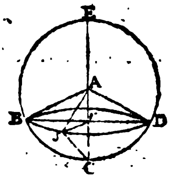
SCHOLIUM 1. Whatever has been demonſtrated of the proportionality of pyramids, priſms, or cylinders, holds equally true of cones, theſe being a third of the latter.

2. It is alſo to be obſerved, that ſimilar cones and cylinders are to each other as the cubes of their altitudes,

or the diamaters of their baſes; the term like ſides being here inapplicable.

PROP. XXII. THEOREM.

If a ſphere be cut by a plane the ſection will be a circle.



Let the ſphere EBD be cut by the plane B*s*D; then will B*s*D be a circle.

For let the planes ABC, A*s*C paſs through the axis of the ſphere EC, and be perpendicular to the plane B*s*D.

Alſo draw the line BD; and join the points A, D and *r*, *s*:

Then ſince each of theſe planes are perpendicular to the plane B*s*D, their common ſection A*r* will alſo be perpendicular to that plane (VII. 14.)

And, becauſe the ſides AB, A*r*, of the triangle AB*r* are equal to the ſides A*s*, A*r* of the triangle A*sr*, and the angles A*r*B, A*rs* are right angles, the ſide *r*B will be equal to the ſide *rs* (I. 4.)

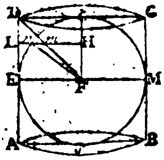
In like manner, the ſides AD, A*r*, of the triangle AD*r*, being equal to the ſides A*s*, A*r*, of the triangle A*sr*, and the angles A*r*D, A*rs* right angles, the ſide *r*D will alſo be equal to the ſide *rs* (I. 4.)

The lines *r*B, *r*D and *rs*, are, therefore, all equal; and the ſame may be ſhewn of any other lines, drawn from the point *r* to the circumference of the ſection; whence B*s*D is a circle, as was to be ſhewn.

COR. The centre of every ſection of a ſphere is always in a diameter of the ſphere.

PROP. XXIII. THEOREM.

Every ſphere is two thirds of its circumſcribing cylinder.



Let *r*E*s*M be a ſphere, and DABC its circumſcribing cylinder; then will *r*E*s*M be two thirds of DABC.

For let AC be a ſection of the ſphere through its centre F; and parallel to DC, or AB, the baſe of the cylinder, draw the plane LH, cutting the former in *n* and *m*; and join FE, F*n*, FD and F*r*.

Then, if the ſquare E*r* be conceived to revolve round the fixed axis F*r*, it will generate the cylinder EC; the quadrant FE*r* will alſo generate the hemiſphere EM*r*E; and the triangle FD*r* the cone FDC.

And ſince FH*n* is a right angled triangle, and FH is equal to H*m*, the ſquares of FH, H*n*, or of H*m*, H*n*, are equal to the ſquare of F*n*.

But F*n* is alſo equal to FE or HL; whence the ſquares of H*m*, H*n* are equal to the ſquare of HL: or the circular ſections whoſe radii are H*m*, H*n* are equal to the circular ſection whoſe radius is HL (VIII. 5. *Cor.*)

And as this is always the caſe, in every parallel poſition of HL, the cone FDC and cylinder EC, which are compoſed of the former of theſe ſections, are equal to the hemiſphere EM*r*E, which is compoſed of the latter.

But the cone FDC is a third part of the cylinder EC (VIII. 21.); whence the hemiſphere EM*r*E is equal to the remaining two thirds; or the whole ſphere *r*E*s*M is equal to two thirds of the whole cylinder DABC, as was to be ſhewn.

COR. 1. A cone, hemiſphere, and cylinder, of the ſame baſe and altitude, are to each other as the numbers 1, 2, and 3.

COR. 2. All ſpheres are to each other as the cubes of their diameters; theſe being like parts of their circumſcribing cylinders.

NOTES AND OBSERVATIONS.

DEF. 1. BOOK I.

THE definition of a ſolid, contrary to the uſual method, is here made the firſt of the firſt Book; as thoſe of a point, line and ſuperficies are all derived from it, and cannot be underſtood without it. EUCLID ſeems to have placed it in the eleventh book of the Elements, for the ſake of uniformity; but arrangements of this kind, which are merely arbitrary, are but of little conſequence, and ſhould therefore always be made to give place to perſpicuity and the natural order of things.

DEF. 2, 3, 4. BOOK I.

Theſe definitions are now, by means of the former, rendered perfectly clear and intelligible, ſo that any farther elucidation of them is altogether unneceſſary. DR. SIMSON has endeavoured to ſhew, by a formal proof, drawn from the conſideration of a ſolid, that a point, according to EUCLID's definition, is without parts, a line without breadth, and a ſurface without thickneſs; but this, and all other demonſtrations of the ſame kind, are unſcientific and ſuperfluous; for theſe properties are ſo obviouſly eſſential to the things defined, that they cannot, even in idea, be ſeparated from them. If a point had parts, it would be a line; if a line had breadth it would be a ſuperficies; and if a ſuperficies had thickneſs, it would be a ſolid; which are all manifeſt contradictions. It is, beſides, a ſure ſign that a definition is badly expreſſed, when it requires a number of prolix arguments to eſtabliſh its truth and propriety.

DEF. 5. BOOK I.

EUCLID's definition of a right line is not expreſſed in ſo accurate and ſcientific a manner as could be wiſhed; the *lying evenly* between its extreme points, is too vague and indefinite a term to be uſed in a ſcience ſo much celebrated for its ſtrictneſs and ſimplicity as Geometry. ARCHIMEDES defines it to be the ſhorteſt diſtance between any two points; but this is equally exceptionable, on account of the uncertain ſignification of the word *diſtance*, which, in common language, admits of various meanings. That which is here given, is, perhaps, not much preferable to either of theſe. The term right, or ſtraight line, is, indeed, ſo common and ſimple, that it ſeems to convey its own meaning, in a more clear and ſatisfactory manner than any explanation which can be given of it. DR. AUSTIN, in his Examination of the firſt ſix books of the Elements, propoſes a ſingular emendation of this definition, which includes the conſideration of right lines, inſtead of a right line, as the caſe manifeſtly requires.

DEF. 6. BOOK I.

Some call a plane ſuperficies that which is the leaſt of all thoſe having the ſame bounds: and others, that which is generated by the motion of a right line, not moving in the direction of itſelf; but theſe definitions are too complex and obſcure to anſwer the purpoſe required. EUCLID defines it to be that which *lies evenly* between its lines; which is liable to the ſame exception as that given of a right line: nor is the one which has been ſubſtituted in the place of this, by DR. SIMSON, and other Editors, ſo ſimple and perſpicuous as could be wiſhed. Nothing is gained by the explanation of a term, if the words in which it is expreſſed are equally, or more, ambiguous, than the term itſelf: for this reaſon, that which is here given, has been preferred to either of thoſe abovementioned: though, perhaps, it may not be equally commodious in certain caſes.

It is alſo to be remarked, that EUCLID never defines one thing by the intervention of another, as is the caſe in DR. SIMSON's emendation; ſo that if this method had occurred to him, he would certainly have rejected it.

DEF. 7. BOOK I.

The general definition of an angle in EUCLID, has been properly objected to, by ſeveral of the modern Editors, as being unneceſſary, and conveying no diſtinct meaning; and in DR. SIMSON's emendation of the ninth, there ſeems to be ſtill a ſuperfluous condition. He defines a rectilineal angle, to be "the inclination of two ſtraight lines to one another, which meet together, but are not in the ſame ſtraight line." Now their not being in the ſame ſtraight line, is a neceſſary conſequence, obviouſly included in their having an inclination to each other; and, therefore, to make this an eſſential part of the definition, is certainly improper, and unſcientific.

DEF. 8, 9. BOOK I.

EUCLID includes a right angle and a perpendicular in the ſame definition, which appears to be immethodical, and contrary to his uſual cuſtom. They are certainly diſtinct things, though dependent upon each other; and have as much claim to be ſeparately defined, as a circle and its diameter.

DEF. 15. BOOK I.

The definition of a circle from its generation, has been thought by DR. BARROW and others, to be preferable to EUCLID's, or the one here given; as it is ſuppoſed to furniſh its properties more readily, and to have the ſtill farther advantage of ſhewing the actual exiſtence of ſuch a figure, independent of any hypotheſis, but that of granting the poſſibility of motion. But the requiſition of this poſtulatum, appears to be a ſufficient reaſon why EUCLID rejected ſuch a definition. The principles of pure Geometry, have no dependence upon motion, and it is, therefore, never uſed in the Elements, but in two or three places of the eleventh book, where it could not, without much obſcurity and circumlocution, have been eaſily avoided. It is beſides, neither ſo ſimple, nor convenient to refer to, as EUCLID's; which, in theſe reſpects, is as commodious as could be wiſhed.

DEF. 20. BOOK I.

DR. BARROW, and other writers of conſiderable eminence, have cenſured EUCLID for defining parallel lines, from the negative property of their never meeting each other; and to this they attribute all the perplexity and confuſion, which has hitherto attended this delicate ſubject: affirming it as an utter impoſſibility, that any of the properties of theſe lines, can be derived from a definition which contains only a ſimple negation. But theſe aſſertions appear to be groundleſs; for the definition is founded on one of the moſt familiar, ſimple and obvious properties of parallel lines, which either reaſon or ſcience can diſcover: and, on this account, it is certainly preferable to any other that could have been formed from more abſtruſe and complicated affections of thoſe lines, how ready and uſeful ſoever ſuch a definition might have been found in its application.

The aſſertion, likewiſe, that none of the other properties of parallel lines can be derived from this definition, has been unadviſedly made; for the 27th Prop. of the firſt Element, which is the ſame as the 22d of the preſent performance, is fairly and elegantly demonſtrated by it; and by means ſomething ſimilar to thoſe made uſe of by Dr. Simson, in his Notes upon the 29th Prop. it would not be difficult to ſhew that all the other properties of thoſe lines may be derived from this definition, without the aſſiſtance of the 12th axiom, or any other of the ſame kind. Dr. Simson, indeed, in his attempt to demonſtrate this axiom, has made ſeveral paralogiſms which render his reaſonings altogether invalid, and nugatory. Paſſing by others, of leſs conſequence, it will be ſufficient to obſerve, that in his fifth Prop. he takes it for granted, that a line, which is perpendicular to one of two parallel lines, may be produced till it meets the other: now this is a particular caſe of the very thing he is endeavouring to prove, which is ſo ſtrange an overſight, that it is remarkable how it could eſcape his obſervation.

This, however, is not the only inſtance of an unſucceſsful attempt to prove the truth of the 12th axiom; for Clavius and others have committed ſimilar miſtakes, and Dr. Austin, who has endeavoured to demonſtrate it by means of a new definition of parallel lines, has made uſe of an aſſumption equally unwarrantable with that mentioned above. That the theory of parallel lines, as it is given in the Elements, is very imperfect, cannot be denied; but no one has yet been ſubſtituted in its place which is not equally defective; and in ſome inſtances ſtill more exceptionable: particularly as they are founded on a definition which is derived from an adventitious property of thoſe lines, inſtead of one which is inherent and neceſſary, as the nature of the ſubject requires.

Whether EUCLID was the author of this axiom cannot perhaps, at this time, be eaſily determined; but it is certainly a diſgrace to the Elements. The truth of the property here aſſumed as a thing to be granted, is ſo far from being obvious, that it requires demonſtration as much as any Prop. in the Elements; and it is always obſerved that learners, inſtead of giving that ready aſſent to it which an axiomatical principle requires, receive it with doubt and heſitation, and are ſcarcely able to comprehend the meaning of it. The one which is here made the 4th poſtulate, though nearly the ſame thing in effect, is much more clear and intelligible.

PROP. 1. BOOK I.

In the demonſtration of this propoſition, by EUCLID, that part which relates to the interſection of the circles is, very improperly, omitted; for in a work of this kind, nothing, however evident, ought to be taken for granted; and particularly at the firſt outſet, where a ſtrictneſs of elucidation is peculiarly neceſſary. The paſſing of the circles through each other's centres is, indeed, a ſufficient reaſon why they muſt cut each other; but this ſhould certainly have been mentioned in the demonſtration.

Prop. 2. Book I.

Proclus, and other writers, have obſerved, that this problem admits of ſeveral caſes, according to the ſituation of the point A; but there is only one of them that can properly be called a ſeparate caſe, which is when the point A is at either of the extremities of the given line: and in this caſe, if a circle be deſcribed from the given point, at the diſtance CB, any of the radii of that circle will be the line required. In all other ſituations of the point A, whether in the line AB, or out of it, the conſtruction and demonſtration will be the ſame as that given in the text; which differs from Euclid's only in the producing of the line DA, after the circle FHG is deſcribed; this being thought more conformable to the terms of the propoſition.

Prop. 3. Book I.

In the conſtruction of this problem the line AD may fall upon the line AB, and then the thing required is done. The given lines C and AB may alſo meet each other, at the point A; and then a circle deſcribed from that point, with the radius C, will cut off from AB the part required. This caſe occurs in the conſtruction of the 5th propoſition following, and in ſeveral other parts of the Elements, and, for that reaſon, ought to have been mentioned. In all other poſitions of the two given lines Euclid's conſtruction and demonſtration are general.

Prop. 4. Book I.

The demonſtration of this propoſition has been frequently objected againſt, as being too mechanical. But this complaint is frivolous and ill founded; for the operation of placing one triangle upon the other, is a mental one, and what is to be conſidered as poſſible to be effected, rather than actually done. There is, beſides, no other way in which the equality of theſe figures can be eſtabliſhed, ſo that any cavils about its merits or defects are entirely precluded.

PROP. 5. BOOK I.

EUCLID, in his demonſtration of this propoſition, has ſhewn that the angles below the baſe are, alſo, equal to each other. But as this property is never referred to throughout the Elements, except in the demonſtration of the 2d caſe of the 7th propoſition following, the whole of which is both aukward and unneceſſary, it would have been better to have omitted it, and confined the demonſtration, in the preſent inſtance, to the equality of the angles above the baſe, which is a property much more generally uſeful.

PROP. 6. BOOK I.

The demonſtration of this propoſition, in EUCLID, is immethodical and defective. It is not ſufficient to ſhew that one ſide is not greater than the other, but it ought, alſo, to be ſhewn that it is not leſs, before their equality can be fairly inferred. It is true, indeed, that either of the ſides may be taken at pleaſure, and the ſame thing will follow: but this obſervation ſhould have been made, and then the premiſes, which they do not at preſent, would have authorized the deduction required. The ſame objection may be made to ſeveral other propoſitions in the Elements.

PROP. 7. BOOK I.

This is the ſame as EUCLID's 8th propoſition, but demonſtrated in a different manner, in order that the preceding one, which is altogether uſeleſs, might be omitted. PROCLUS demonſtrates it in nearly the ſame manner; but he makes three caſes of it, when it may be done generally in one; for if the longeſt ſides, or rather thoſe which are not ſhorter than any other, be applied together, there can be no ambiguity in the ſpecies of the triangles.

PROP. 8. BOOK I.

This propoſition is made an axiom by EUCLID; but it is certainly not a truth of that kind: for when two right angles are found in ſeparate and diſtinct figures, there is nothing in the definitions or poſtulates, from which their equality to each other can be fairly inferred.

PROP. 12. BOOK I.

It is not ſhewn by EUCLID, in his demonſtration of this problem, that the circle made uſe of in the conſtruction, will cut the given line in two points, which as much requires proof as Prop. 2. Book III. which is nearly its converſe. For this reaſon the conſtruction given in the text has been preferred; but in a work where the utmoſt ſcientific rigour is required, it would be better to conſtruct the problem without the intervention of the circle, by means of right lines only, which may eaſily be done.

PROP. 13. BOOK I.

In the enunciation of this Prop. by EUCLID, the angles are ſaid to be either equal to two right angles, or together equal to two right angles: but the former part of this ſeems to be unneceſſary; for in all caſes, whether the angles be each of them a right angle, or not, they are together equal to two right angles.

PROP. 16. BOOK I.

As the outward angle of a triangle is afterwards ſhewn to be equal to the two inward oppoſite angles, it is much to be wiſhed that the preſent Prop. which is only a partial caſe of the former, could be removed from the Elements: but this cannot eaſily be done; for the following propoſition, and the firſt relating to parallel lines, are not otherwiſe to be demonſtrated. The next Prop. in EUCLID, is, however, quite unneceſſary, as the firſt place in which it occurs is Prop. 18, B. 3, where a reference may be as readily made to the general propoſition.

PROP. 19. BOOK I.

The demonſtration of this propoſition, as it is given by EUCLID, is extremely defective; for the whole deſign of the problem is to ſhew that of three right lines, under certain ſpecified reſtrictions, a triangle may be formed; and as no uſe whatever is made of theſe reſtrictions, either in the conſtruction or demonſtration, both the arguments adduced, and the concluſions derived from them, are entirely nugatory. This defect was obſerved by MR. SIMPSON, between whom and his antagoniſt DR. SIMSON, it occaſioned ſome controverſy, which drew from the latter ſome very haſty unſcientific expreſſions, not much comporting with the character of ſo ſtrict and accurate a Geometrician.

PROP.

PROP. 28. BOOK I.

In moſt editions of EUCLID, two corollaries are affixed to this propoſition; which are equally or more intricate than the propoſition itſelf. DR. AUSTIN has endeavoured to prove that theſe, and moſt of the other corollaries, to be found in the Elements, were not introduced by EUCLID, but by ſome of his commentators, or interpreters; and there are many reaſons for believing that this opinion is not ill founded. It is generally allowed, that EUCLID wrote a book entitled Corollaries, which were a collection of conſequences deducible from his Elements; and, therefore, it is not to be imagined that they were originally inſerted in that work; as in that caſe it would have been quite unneceſſary to have publiſhed them in a ſeparate performance. Beſides this, the chain of reaſoning is complete without them, as is evident from their being ſeldom referred to in any propoſition. In all caſes, however, where a uſeful truth of this kind can be readily deduced from a preceding demonſtration, there appears to be no impropriety in making it a corollary.

PROP. 31. BOOK I.

This propoſition, as it ſtands in moſt of the early editions, has three diſtinct caſes, which all require to be ſeparately demonſtrated. DR. SIMSON, by changing the mode of demonſtration, has reduced it to two: but by an obvious alteration in the enunciation of the I. 26. EUC. which is the ſame as our 21ſt, the propoſition, both for parallelograms and triangles, might have been demonſtrated generally, in one caſe only; which, when it can be done, is always to be preferred. In this part of the work, alſo, ſeveral other alterations have been made, the reaſon for which will be given in the notes to the ſecond book.

PROP. 33. BOOK I.

This propoſition is ſubſtituted in the place of Prop. 42, 44, and 45 of EUC. B. I. as being leſs intricate, and equally uſeful in its application. One of the principal deſigns of theſe propoſitions, is to ſhew, that a parallelogram, under certain conditions, can be formed; and as this can be more readily effected by other methods, the preference has been given to that which appears the moſt ſimple. It may alſo be obſerved, that the 44th propoſition of EUCLID is not legally demonſtrated; for the parallelogram BF, which makes a part of the conſtruction, cannot be formed from Prop. 42, as is directed, being entirely a different caſe: and as the 45th is derived from the 44th, it muſt alſo be liable to the ſame objection.

PROP. 34, 35. BOOK I.

Theſe propoſitions are delivered by EUCLID in a different form, and not given till the 6th book; but as they are extremely eaſy, and of frequent uſe in their application to other propoſitions, in the preceding books, they have been here introduced as early as poſſible, and demonſtrated independently of the doctrine of proportion; which, it is imagined, beginners will conſider as an advantage, as they ſeldom arrive to ſuch a proficiency, in a knowledge of the Elements, as to obtain clear and ſatisfactory ideas of that intricate ſubject.

PROP.

PROP. 1. BOOK II.

In EUCLID's demonſtration of this problem, it ought to have been proved that the lines which are directed to be drawn parallel to AB, AD, will meet each other; or otherwiſe it is not certain that the ſquare required can be formed. On this account, a mode of conſtruction has been here obſerved, which is not liable to that objection.

PROP. 2, 3, 4. BOOK II.

Theſe propoſitions, which are not in EUCLID, may by ſome, be thought unneceſſary; but they muſt either be demonſtrated, or aſſumed; as the firſt, in particular, is wanted in almoſt every propoſition of the ſecond book; and the others are frequently required in ſeveral parts of the Elements. Why they were omitted by EUCLID does not appear; they are certainly not axiomatical, nor more evident in themſelves than many others which he has ſcrupulouſly demonſtrated.

PROP. 5. BOOK II.

This propoſition is placed in the ſecond book, for the purpoſe of demonſtrating it in a more general manner than has been done by EUCLID; and in order that ſome others, of little importance, might be more eaſily omitted. The demonſtration depends principally upon the firſt propoſition, mentioned above; and this, among other inſtances, is ſufficient to ſhew the utility of that theorem, and the neceſſity of its being introduced into the Elements.

PROP 7. BOOK II.

This theorem, which is not in EUCLID, is given chiefly on account of its application to ſome of the following propoſitions, the demonſtrations of which are, by this means, rendered more conciſe and elegant.

PROP. 13. BOOK II.

All the theorems in EUCLID's ſecond book, which relate to the diviſion of a line into more than two parts, are here omitted, as they are commonly found tedious and embarraſſing to beginners, and are not of any very extenſive uſe. The preſent propoſition, which is not in EUCLID, is much more generally applicable; and this, together with the preceding ones, will be found ſufficient for moſt geometrical purpoſes.

PROP. 16, 19, 20 and 21. BOOK II.

Theſe propoſitions, though not in EUCLID, are frequently wanted, particularly the 1ſt, 2d, and 3d, which are, alſo, equally remarkable, both for their elegance and utility.

PROP. 1. BOOK III.

It is properly obſerved by DR. SIMSON, in his notes upon this book, that the objections which have been uſually made againſt the indirect method of proof, uſed in this and ſeveral other propoſitions in the Elements, are injudicious and ill founded; as it is obvious to every one, who has duly conſidered the ſubject, that there are many things which cannot be proved in any other way. There is, however, a real defect in the demonſtration of this propoſition, that eſcaped his notice; which is, that the fictitious centre, or point G, may be taken in the line EC; and in this caſe the demonſtration given by EUCLID will not hold.

PROP. 4. BOOK III.

This propoſition is the ſame as the 9th of EUCLID, Book III. but demonſtrated in a manner which it is imagined will appear ſomething more clear and ſatisfactory, at leaſt to beginners. According to his method the propoſition admits of ſeveral caſes; and in that which he has choſen as a general one, the fictitious centre, or point E, is ſo taken, that the proof would be exactly the ſame for two equal right lines as for three, which is a manifeſt imperfection.

PROP. 5. BOOK III.

EUCLID has given this theorem in his 3d Book, in the form of a definition; which is the more remarkable, as he appears, in ſeveral parts of the Elements, to be well aware, that the equality of no two figures can be admitted but from the teſt which he has laid down in the 8th axiom.

PROP. 6, 7. BOOK III.

Theſe theorems are, in ſubſtance, the ſame as EUCLID's, but differently enunciated, in order to accommodate beginners, who are generally embarraſſed with the aukwardneſs of the figures, and the two fictitious centres in the laſt propoſition; the latter of which are here avoided. It may alſo be obſerved that the demonſtrations of theſe propoſitions are not ſtrictly ſcientific; ſince, for aught that appears to the contrary, the circles may touch each other in more points than one, in which caſe the proof he has given would be nugatory. To avoid this, the ſucceeding propoſitions ſhould have been placed prior in order to the preſent ones, and demonſtrated independently of them; which, however, cannot eaſily be done. For this reaſon, and to avoid as much as poſſible all theorems which are otherwiſe of little importance, the touching of the circles in one point only, has been here inferred from the definition. A ſimilar objection may, likewiſe, be made againſt the 5th, 6th, and 10th theorems of EUCLID, Book III; the laſt of which ſhould have been demonſtrated firſt; for, as they now ſtand, ſeveral things which require proof, as much as the propoſitions themſelves, are taken for granted.

PROP. 10. BOOK III.

In this propoſition, no mention is made of the cornicular angle, or that which is ſuppoſed to be formed by the circumference and tangent, at the point of contact; as it is of no uſe whatever in Geometry, and ought never to have been admitted into the Elements. DR. SIMSON ſuſpects, with VIETA, that it is an interpolation, and on that account has properly rejected it; but there are ſtill ſome particulars, in his enunciation of this propoſition, which appear to be equally unneceſſary. The theorem is, therefore, here propoſed in as ſimple a manner as poſſible, and reſtricted to that caſe which moſt frequently occurs.

PROP. 14. BOOK III.

In EUCLID's demonſtration of the ſecond caſe of this theorem, the following propoſition has been taken for granted, viz. "If one magnitude be double of another, and a part taken from the firſt, be double of a part taken from the ſecond, the remainder of the firſt will be double the remainder of the ſecond." But as this aſſumption, which has hitherto been tacitly acquieſced in, is not derived from the axioms, or any thing which has been previouſly demonſtrated, it is certainly improper, and unjuſtifiable. In order, therefore, to render the demonſtration of this caſe more ſtrict and ſcientific, it is here given in a different manner, which is equally eaſy with the former, and not liable to the ſame objection.

PROP. 15. BOOK III.

DR. AUSTIN in his examination of the firſt ſix books of the Elements, is of opinion that the ſecond caſe of this propoſition, which has been added by DR. SIMPSON, and other modern Editors, is unneceſſary. "The former propoſition, he obſerves, is general; and, therefore, it is immaterial, whether the part of the circumference upon which the angles at the centre and circumference ſtand, be greater or leſs than a ſemicircle." But this obſervation is foreign to the purpoſe; for as the arc which ſubtends an angle at the centre, muſt always be leſs than a ſemicircle, no ſuch angle can be introduced into the conſtruction of this caſe; and therefore the demonſtration of it muſt neceſſarily be obtained in ſome way different from the former.

PROP. 18, 19. BOOK III.

The firſt of theſe propoſitions is the ſame in effect as the 25th of EUCLID, Book III, but ſomething more ſimple, being demonſtrated generally in one caſe. The other is the converſe of our 17th, or EUCLID's 22d, which he has omitted, but for what reaſon does not appear, as it is of frequent uſe in its application.

PROP. 20. BOOK III.

This propoſition differs from the 24th of EUCLID, Book III, only in the enunciation, which was done to avoid the neceſſity of defining ſimilar ſegments of circles. For as this definition, which is nothing more than EUCLID's 21ſt propoſition, in another form, cannot poſſibly be underſtood, till it is ſhewn that all angles in the ſame ſegment are equal to each other, it is altogether uſeleſs, and contrary to the nature of a definition, which requires that it ſhould be expreſſed in ſuch terms, and derived from ſuch properties as are ſimple and obvious.

PROP. 27, 28, 29. BOOK III.

The demonſtrations of theſe propoſitions are confined to one caſe, which, though it does not include every poſſible poſition of the lines, will, it is conceived, be thought ſufficiently general. EUCLID, in this inſtance, is much more particular; having ſcrupulouſly demonſtrated caſes of leſs moment than many others in the Elements which are taken for granted. And the ſame want of uniformity may be obſerved in ſeveral other propoſitions, with reſpect to the ſtrictneſs or laxity of their demonſtrations.

PROP. 3. BOOK IV.

DR. SIMSON in his note upon EUCLID, Prop. 5, B. 4, obſerves "that the demonſtration of this problem, has been ſpoiled by ſome unſkilful hand. For he does not demonſtrate, as is neceſſary, that the two ſtraight lines, which biſect the ſides of the triangle, at right angles, muſt meet one another." After which, it appears ſomething ſingular, that this able Geometrician ſhould not perceive that a ſimilar omiſſion had been made in the demonſtration of the 3d, and ſeveral other propoſitions of this book; in which the neceſſity of proving that certain lines will meet each other is equally obvious. In all theſe caſes, therefore, that part of the demonſtration is now ſupplied, and the different ſolutions, by that means, rendered more complete.

PROP. 10. BOOK IV.

This propoſition has been purpoſely altered from EUCLID, in order to render the conſtruction of the following one more practical and ſimple. It is now, alſo, properly limited, which EUCLID's is not; for according to his enunciation of the problem, an infinite number of triangles may be formed, which will anſwer the conditions required. The ſame objection is likewiſe applicable to ſeveral other propoſitions in the Elements; and though it may, to ſome, appear trifling, it is certainly a departure from that ſtrictneſs and preciſion which, in a work of this nature, are generally conſidered as eſſential requiſites. An inſtance of this kind occurs even in the 2d Prop. of B. I, which, however, is not ſo eaſily remedied.

PROP. 11. BOOK IV.

Some of the Commentators have obſerved, that EUCLID's method of inſcribing a pentagon in a circle, is much leſs ſimple and elegant than that of PTOLEMY in the 1ſt Book of his Almageſt; and for that reaſon think it ought to have been given in the Elements. Whilſt others maintain that the demonſtration of PTOLEMY's conſtruction depends wholly on the 13th Book, and that, therefore, if EUCLID had known it, he could not have inſerted it in the preſent Book; the materials for it not being yet prepared. This, however, is not true; for it has been clearly ſhewn by the author, in a periodical publication for the year 1786, that the truth of this conſtruction may be proved by means of the firſt three Books of the Elements only; but the reaſon why it has not been given in the text is, that the demonſtration, being ſomething more intricate, might not have been ſo readily comprehended by beginners, for whoſe uſe this work is principally deſigned.

DEF. 5. BOOK V.

This definition, which is the ſame in effect as that given by EUCLID, has been the occaſion of much controverſy and diſpute among Mathematicians; many of them thinking it foreign to the purpoſe, and others too difficult and obſcure to be made the leading principle of a doctrine ſo uſeful and neceſſary as that of proportion. But, from a mature conſideration of the ſubject, it is ſufficiently evident that no other definition, equally applicable and general, could have been given; and, therefore, the neceſſity of the caſe required that the preſent one ſhould be adopted in preference to all others.

That it is not ſo ſimple and evident as that which may be given of numbers, or commenſurable magnitudes, cannot be denied; but this ariſes from the nature of the ſubject, and is not to be avoided. There are ſome proportional magnitudes, ſuch, for inſtance, as the ſide of a ſquare and its diagonal, which have no common meaſure, and conſequently cannot be defined by that means. Some other definition was, therefore, to be found, which would hold in this, and all other caſes, without exception; and as the one in queſtion anſwers theſe conditions, and is, at the ſame time, equally commodious in practice, nothing farther can be expected.

In order, however, to accommodate learners, who are ſeldom able to comprehend EUCLID's 5th Book, ſuch alterations have been made in this part of the work, as it is preſumed will render it much more clear and intelligible. Among others, the definition above-mentioned is more conciſely enunciated; by which it is made to appear leſs intricate and involved; and conſequently may be more eaſily remembered and applied. Every thing which relates to a greater and leſs ratio is alſo rejected, as being obſcure and unneceſſary. And as brevity was here thought particularly requiſite, ſuch propoſitions only have been introduced, as are obviouſly uſeful; the reſt being conſidered as impediments in the way of the learner, and, for that reaſon, unfit for an elementary performance, whoſe principal aim ſhould be clearneſs and perſpicuity.

For a further account of the doctrine of ratios, as delivered by EUCLID, the reader is referred to the 7th and 8

8th of Dr. Barrow's Mathematical Lectures, where the uſual objections which are made to this method are fully refuted.

Prop. 2. Book V.

This Prop. is the ſame as the Cor, to Euclid's 2d Prop. of Book V, which Dr. Simson has marked with inverted commas, as being unneceſſary. But, whoever conſiders this Book with attention, will obſerve, that the corollary is much more uſeful and general than the propoſition. It is, indeed, ſtrictly ſpeaking, no corollary to the propoſition in queſtion, and for that reaſon is properly enough diſcarded; but it would have been much better to have ſtruck out the propoſition, and ſubſtituted the corollary in its place.

Prop. 4. Book V.

This propoſition, which is required in the demonſtration of ſome of the following ones, is not expreſly enunciated by Euclid, being introduced into the demonſtration of the 8th propoſition, without any farther notice. But as it is a diſtinct theorem, the truth of which is much leſs obvious than that of ſeveral others in this Book, it ought to have been ſeparately demonſtrated; and particularly as it is of conſiderable uſe in its application.

Prop. 16, 17. Book V.

It has been properly obſerved, by Mr. Simpson, that the manner in which the compoſition and diviſion of ratios is treated of by Euclid, is defective, as not being ſufficiently general. It is alſo commonly found very abſtruſe and embarraſſing to beginners, on account of the complicated terms in which it is enunciated, and the number of caſes to be ſeparately demonſtrated. For theſe reaſons, it was deemed neceſſary to give the propoſitions a more ſimple and general form, and to render the demonſtrations of them as conciſe and perſpicuous as poſſible. The 17th, in ſtrictneſs, has two diſtinct caſes; but as the ſecond differs from the firſt only by inverting the terms, it was judged ſufficient to mention it in a Scholium.

DEF. 1. BOOK VI.

DR. AUSTIN objects to this definition, becauſe it does not yet appear that any rectilineal figures can have their angles equal, each to each, and the ſides about them proportional. He would therefore define ſimilar triangles firſt, which may be done only from the equality of their angles; and after this to call thoſe ſimilar rectilineal figures, of more than three ſides, which conſiſt of an equal number of ſimilar triangles, ſimilarly ſituated. This is no doubt an alteration which might prove advantageous to learners; but as it appears illogical, and contrary to the method made uſe of in other caſes of a like nature, the original definition was thought preferable.

PROP. 12, 13, 16, 17. BOOK VI.

Several alterations have been made in the demonſtrations of theſe propoſitions, as they were given by EUCLID, with the view of rendering them more ſimple and eaſy; but as they are in general ſuch as may be readily diſcovered by inſpection, or by comparing the theorems with thoſe in the Elements, it will be unneceſſary to point them out to the reader.

PROP. 19, 20, 22, 23. BOOK VI.

Theſe propoſitions are not in EUCLID, though their utility and elegance certainly entitled them to a place in the Elements. The latter, in particular, may ſupply the place of the 27th, 28th, and 29th of EUCLID; which are certainly very awkward propoſitions, and are ſeldom well underſtood by beginners. The uſe which was made of them by ſome of the ancient Geometers, has been urged as a ſufficient reaſon why they ought to be retained in the Elements; but, upon this principle, numberleſs other propoſitions might be inſerted, which would ſwell this compendious and beautiful ſyſtem into a large volume; and make it appear more like a common place book than a ſimple regular performance, judiciouſly arranged in all its parts, and diſplaying only the firſt and moſt important principles of the ſcience.

PROP. 25, 26, 27. BOOK VI.

Theſe elegant and uſeful theorems were not originally in EUCLID; but have been inſerted in ſome of the late editions, by DR. SIMSON and other writers. MR. SIMPSON has given them in the third book of his Elements, and demonſtrated them independently of proportion; which he conceives to be an alteration much for the better: but this will ſcarcely be allowed, when it is conſidered that he was obliged to introduce a new theorem for this purpoſe, manifeſtly derived from the principles laid down in the 5th Book; and that the advantages attending this method, are entirely deſtroyed by a forced and unnatural arrangement.

PROP. 2. BOOK VII.

DR. SIMSON, in his notes upon this propoſition, obſerves, that the enunciation of it, in moſt editions, has been changed and vitiated, by its being propoſed to ſhew that every patt of a triangle is in the ſame plane, inſtead of the way in which it now ſtands; as the property here alluded to is ſaid to be contained in the definition of a triangle, and is conſtantly taken for granted in every part of the firſt ſix books. But he did not perceive that a ſimilar objection might be made to the preceding propoſition; in which it is proved that one part of a right line cannot be in a plane, and another part above it; this being, in like manner, a neceſſary conſequence of the definition he has given of a plane, or rather the very property from which it is derived.

PROP. 3. BOOK VII.

The demonſtration of this theorem, which is new, and more conciſe than that given by EUCLID, is derived from a property of the iſoſceles triangle, not mentioned in the Elements, but which is found of conſiderable uſe in its application to other propoſitions. Several other alterations have alſo been made in different parts of this book, of which, as they will be readily diſcovered by thoſe who are verſed in the ſubject, any farther account is unneceſſary.

BOOK VIII.

The method here followed, of treating the ſolids, is not materially different from that which has been employed by other writers upon the like occaſion; but the propoſitions, it is preſumed, will be found much better arranged, and the ſubject rendered more eaſy and familiar than has hitherto been done. The author however, is well aware that it is liable to critical objections; and had it not been incompatible with the plan of the performance, he certainly would have given it a more ſcientific form; which even from the ſimple principles here employed, might eaſily have been done. But as the advantages attending the preſent mode of demonſtration, eſpecially to learners, appeared ſuperior to every other conſideration, it was adapted in preference to that of EUCLID; which, though more accurate, is frequently found to be tedious and obſcure.

FINIS.

Publiſhed by the ſame AUTHOR.

1. The SCHOLAR'S GUIDE to ARITHMETIC, 7th Edit. 1802. Price 2s. 6d.

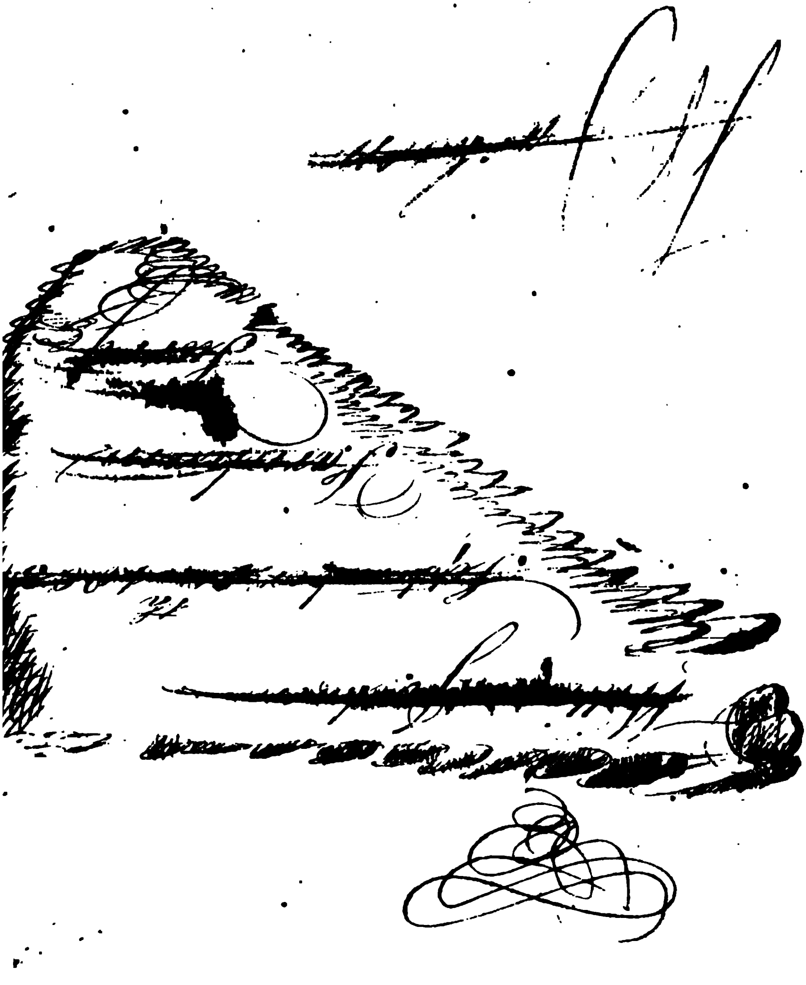
2. An INTRODUCTION to MENSURATION and PRACTICAL GEOMETRY, 7th Ed. Pr. 4s.

3. An INTRODUCTION to ALGEBRA, 6th Ed. Pr. 3s. 6d.

4. An INTRODUCTION to ASTRONOMY, 4th Ed. Pr. 8s. in boards.

Printed for J. JOHNSON, No. 72, St. Paul's Church-Yard.

Bye and Law, Printers, St. John's Square, Clerkenwell.



[illegible]

[illegible]

[illegible] th [illegible]

[illegible]

[illegible]

[illegible]